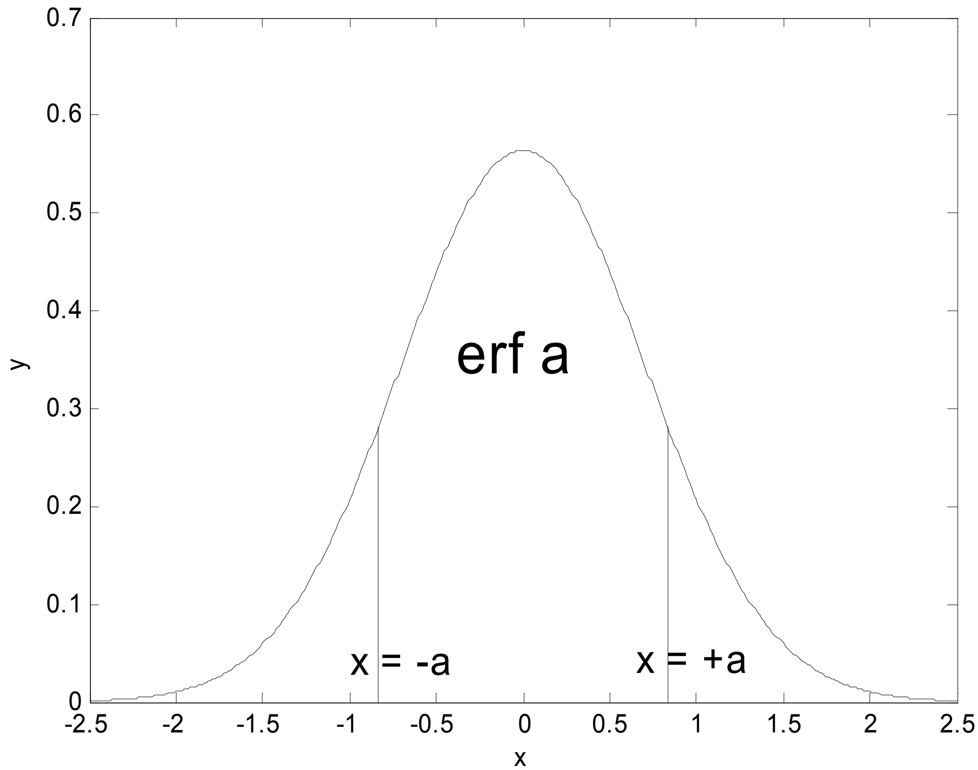
CHAPTER 4
THERMAL CONDUCTION

4.0 *The Error Function*

Before we start this chapter, let's just make sure that we are familiar with the *error function* erf *a*. We may need it during this chapter.

Here is a graph of the gaussian function $y = \frac{1}{\sqrt{\pi}} e^{-x^2}$. 4.0.1



I have chosen the coefficient $1/\sqrt{\pi}$ so that the area under the curve, from $-\infty$ to $+\infty$ is 1. The maximum value, which occurs at $x = 0$, is $1/\sqrt{\pi} = 0.5642$, and it is easy to show that the half width at half the maximum is $\sqrt{\ln 2} = 0.8326$. Also of some interest (though not particularly in this chapter) is the square root of the second moment of area around the *y*-axis. In a mechanical context this would be called the *radius of gyration*. In a statistical context it would be called the *standard deviation*. Either way, its value is $1/\sqrt{2} = 0.7071$. We shall meet the gaussian function again in Chapter 6.

In the present chapter we shall need to make use of the *error function* erf a. This is the area under the gaussian curve from $x = -a$ to $x = +a$:

$$\mathrm{erf}\, a = \frac{1}{\sqrt{\pi}} \int_{-a}^{+a} e^{-x^2}\, dx. \qquad 4.0.2$$

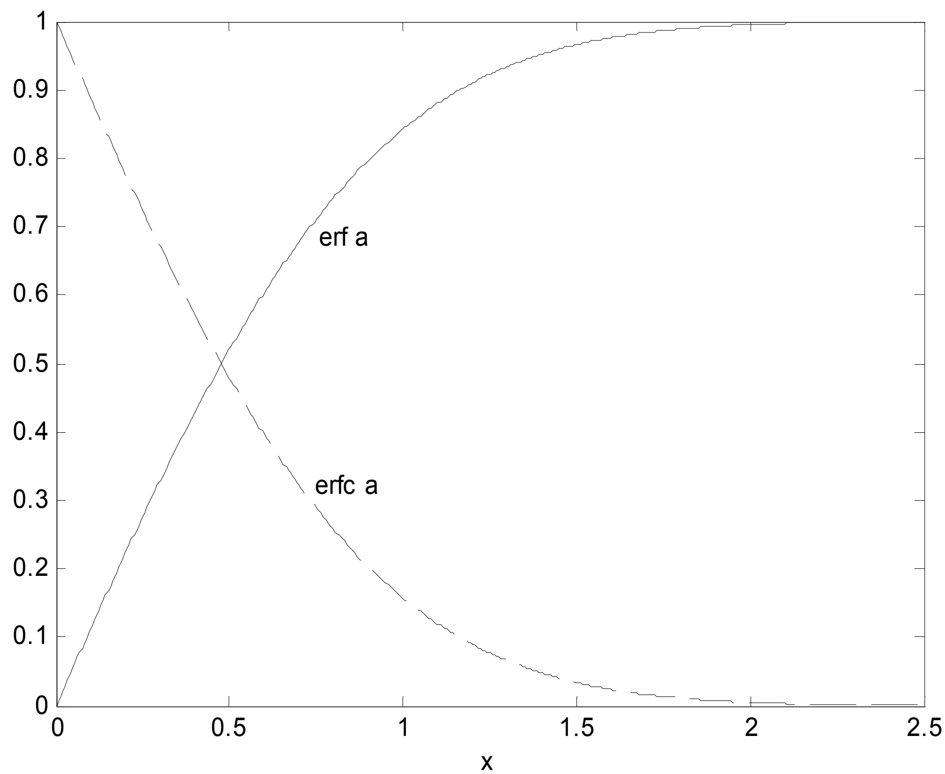
The area outside the limits $x = \pm 1$, which is the area under the two "tails" of the gaussian function, is sometimes called the *complementary error function*:

$$\mathrm{erfc}\, a = 1 - \mathrm{erf}\, a. \qquad 4.0.3$$

It will be clear that erf a goes from 0 to 1 as a goes from 0 to infinity. Note also that

erfc (one standard deviation) = 0.3173
erfc (two standard deviations) = 0.0455.

Here are graphs of erf a (continuous line) and erfc a (dashed line) versus a.



4.1 *Introduction*

While the subject of thermal conduction is an important one, and obviously a proper topic in the theory of heat, it is not really part of the great logical structure of *thermodynamics*, not does it require a wide or deep knowledge of thermodynamics to understand it, at least at an introductory level. In other words, this chapter is more or less a stand-alone chapter. It is not necessary to understand earlier chapters to understand this one; nor, if your primary interest is in thermodynamics, is it necessary to understand this chapter before proceeding to later ones. That is – if you wish − you can skip this chapter without compromising your understanding of any later ones.

4.2 *Thermal Conductivity*

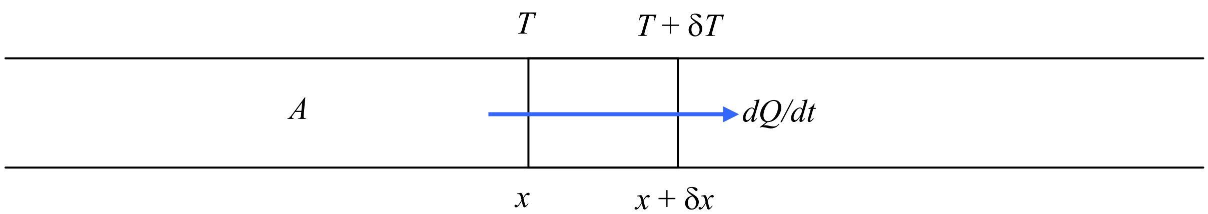


FIGURE IV.1

Figure IV.1 shows heat flowing at a rate dQ/dt along a bar of cross-sectional area A of material. There is a temperature gradient along the length of the bar (which is why heat is flowing down it). At a distance x from the end of the bar the temperature is T; at a distance $x + \delta x$ it is $T + \delta T$. Note that, if heat is flowing in the positive direction as shown, δT must be negative. That is, it is cooler towards the right hand end of the bar. The temperature gradient dT/dx is negative. Heat flows in the opposite direction to the temperature gradient.

The ratio of the rate of heat flow per unit area to the negative of the temperature gradient is called the *thermal conductivity* of the material:

$$\frac{dQ}{dt} = -KA\frac{dT}{dx}. \qquad 4.2.1$$

I am using the symbol K for thermal conductivity. Other symbols often seen are k or λ. Its SI unit is W m^{-1} K^{-1}.

I have defined it in a one-dimensional situation and for an isotropic medium, in which case the heat flow is opposite to the temperature gradient. One can imagine that, in an anisotropic medium, the rate of heat flow and the temperature gradient may be different parallel to the different crystallographic axes. In that case the heat flow and the temperature gradient may not be strictly antiparallel, and the thermal conductivity is a tensor quantity. Such a situation will not concern us in this chapter.

If, in our one-dimensional example, there is no escape of heat from the sides of the bar, then the rate of flow of heat along the bar must be the same all along the bar, which means that the temperature gradient is uniform along the length of the wire. It may be easier to imagine no heat loss from the sides than to achieve it in practice. If the bar were situated in a vacuum, there would be no loss by conduction or convection, and if the bar were very shiny, there would be little loss by radiation.

Order-of-magnitude values of the thermal conductivities of common substances are

Air	0.03	$\mathrm{W\ m^{-1}\ K^{-1}}$
Water	0.6	
Glass	0.8	
Fe	80	
Al	240	
Cu	400	

It is easy to imagine how heat may be conducted along a solid, with the vibrations of the atoms at one end of the solid being transmitted to the next atoms by one atom nudging the next, and so on. However, it is evident from the table, and in any case is common knowledge, that some substances (metals) conduct heat much better than others. Indeed, among the metals, there is a close correlation between the thermal and electrical conductivities (at a given temperature). This suggests that the mechanism for thermal conductivity in metals is the same as for electrical conductivity. Heat is conducted in a metal primarily by electrons.

It would be an interesting exercise to find, from the Web or from other references, the thermal and electrical conductivities of a number of metals. It may be found that thermal conductivities, K, are sometimes quoted in unfamiliar "practical" units, such as BTU per hour per square foot for a temperature gradient of 1 F° per inch, and converting these to SI units of $\mathrm{W\ m^{-1}\ K^{-1}}$ might be a bit of a challenge. Electrical conductivities, σ, decrease somewhat with rising temperature (so do thermal conductivities, but rather less so), so it would be important to find them all at the same temperature. Then you could see whether the ratio K/σ is indeed the same for all metals at a given temperature. This is known as the *Wiedemann-Franz Law.* First-order theory (which we do not give here) predicts that

$$\frac{K}{\sigma T} = \frac{1}{3}\left(\frac{\pi k}{e}\right)^2 = 2.44 \times 10^{-8}\ \mathrm{W\ \Omega\ K^{-1}}. \qquad 4.2.2$$

Here k is Boltzmann's constant and e is the electronic charge. This prediction is found to be obeyed well at room temperatures and higher, but at low temperatures the electrical conductivity increases rapidly with lowering temperature, and the ratio starts to fall well below the value predicted by equation 4.2.2, approaching zero at 0 K.

The reader may be familiar with the following terms in electricity

Conductivity σ
Conductance G
Resistivity ρ
Resistance R

They are related by $G = 1/R$, $\sigma = 1/\rho$, $R = \rho l/A$, $G = \sigma A/l$,

where l and A are the length and cross-sectional area of the conductor. The reader probably also knows that resistances add in series and conductances add in parallel.

We may define some analagous quantities related to heat flow. Thus resistivity is the reciprocal of conductivity, resistance is l/A times resistivity, conductance is A/l times conductivity, and so on. These concepts may come in useful in the following genre of problems beloved of examiners.

A room has walls of area A_1, thickness d_1, thermal conductivity K_1, a door of area A_2, thickness d_2, thermal conductivity K_2, and a window of area A_3, thickness d_3, thermal conductivity K_3, The temperature inside is T_1 and the temperature outside is T_2. What is the rate of heat loss from the room?

We have three conductances in parallel: $\dfrac{K_1 A_1}{d_1}$, $\dfrac{K_2 A_2}{d_2}$ and $\dfrac{K_3 A_3}{d_3}$, and so we have

$$\frac{dQ}{dt} = \left(\frac{K_1 A_1}{d_1} + \frac{K_2 A_2}{d_2} + \frac{K_3 A_3}{d_3}\right)(T_2 - T_1). \qquad 4.2.3$$

Of course, the problem need not be exactly like that. Perhaps you are given the rate of heat loss and asked to find the area of the window. But you get the general idea, and you can probably concoct a few examples yourself. The rate of heat flow is analogous to the current, and the temperature difference is like the EMF of a battery.

4.3 *The Heat Conduction Equation*

The situation described in Section 4.2 and in figure IV.1 was a *steady-state* situation, in which the temperature was a function of x but not of time. We are now going to consider a more general situation in which the temperature may vary in time as well as in space.

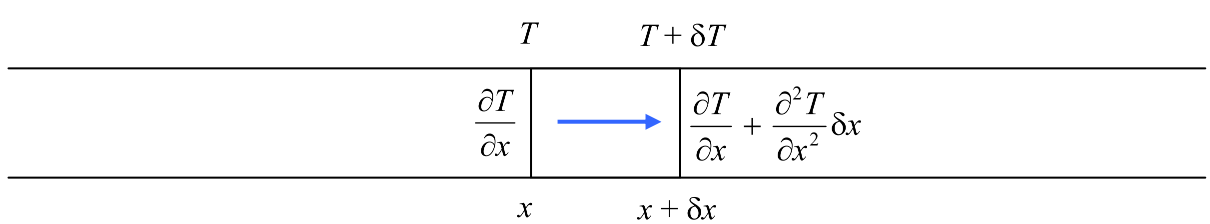


FIGURE IV.3

In this case the temperature gradient is written as a partial derivative, $\frac{\partial T}{\partial x}$ and is not uniform down the length of the rod. We'll suppose it is $\frac{\partial T}{\partial x}$ at x and $\frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial x^2}\delta x$ at $x + \delta x$.

Consider the heat flow into and out of the portion between x and $x + \delta x$. The rate of flow into this portion at x is $-KA\frac{\partial T}{\partial x}$, and the rate of flow out at $x + \delta x$ is $-KA\left(\frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial x^2}\delta x\right)$, so that the net flow of heat into that portion is $KA\frac{\partial^2 T}{\partial x^2}\delta x$. This must be equal to $C\rho A\,\delta x\frac{\partial T}{\partial t}$, where ρ is the density (and hence $\rho A\delta x$ is the mass of the portion), and C is the specific heat capacity.

Therefore
$$C\rho\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2}. \qquad 4.3.1$$

This can be written
$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}, \qquad 4.3.2$$

where
$$D = \frac{K}{C\rho} \qquad 4.3.4$$

is the *thermal diffusivity* ($\mathrm{m^2\ s^{-1}}$).

Equation 4.3.2 is the *heat conduction equation.* In three dimensions it is easy to show that it becomes

$$\dot{T} = D\nabla^2 T. \qquad 4.3.5$$

4.4 *A Solution of the Heat Conduction Equation*

Methods of solving the heat conduction equation are commonly given in courses on partial differential equations. Here we shall look at a simple one-dimensional example.

A long copper bar is initially at a uniform temperature of 0°C. At time $t = 0$, the left hand end of it is heated to 100°C. Draw graphs of temperature versus distance x from the hot end of the bar (up to $x = 100$ cm) at $t = 4$, 16, 64, 256 and 1024 seconds. Draw also a graph of temperature versus time at $x = 5$ cm, up to 1024 seconds. Assume no heat is lost from the sides of the bar.

Data for copper:
$K = 400\ \mathrm{W\ m^{-1}\ K^{-1}}$
$C = 395\ \mathrm{J\ kg^{-1}\ K^{-1}}$
$\rho = 8900\ \mathrm{kg\ m^{-3}}$

whence $D = 1.137 \times 10^{-4}\ \mathrm{m^2\ s^{-1}}$

The equation to be solved is $$D\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}. \qquad 4.4.1$$

From the form of this equation, it is obvious (once it has been pointed out!) that a solution could be found in which $T(x, t)$ is solely a function of x^2/t, or, for that matter, $x/t^{1/2}$.

Thus, let $$u = x/t^{1/2}, \qquad 4.4.2$$

and you will see that equation 4.4.1 reduces to the second order total differential equation

$$D\frac{d^2 T}{du^2} = -\frac{u}{2}\frac{dT}{du}. \qquad 4.4.3$$

Let $T' = dT/du$, and it becomes even easier – a first order equation:

$$D\frac{dT'}{du} = -\tfrac{1}{2}uT'. \qquad 4.4.4$$

Upon integration, we obtain

$$\ln T' = -\frac{u^2}{4D} + \ln A, \qquad 4.4.5$$

where ln A is an integration constant, to be determined by the initial and boundary conditions. (What are the dimensions of A?)

That is, $$T' = A\exp\left[-u^2/(4D)\right]. \qquad 4.4.6$$

We have to integrate again, with respect to u:

$$T = A\int \exp\left[-u^2/(4D)\right]du. \qquad 4.4.7$$

Now, $T = 100$ °C at $x = 0$ for any $t > 0$. That is, $T = 100$ for $u = 0$.

And $T = 0$ °C at $t = 0$ for any $x > 0$. That is, $T = 0$ for $u = \infty$.

Therefore $$100 - 0 = A\int_\infty^0 \exp\left[-u^2/(4D)\right]du. \qquad 4.4.8$$

The integral is slightly difficult though well known. I'll just state the answer here; it is $-\sqrt{\pi D}$. From this, we find that the integration constant is

$$A = -5284 \text{ K m}^{-1}\text{ s}^{1/2}. \qquad 4.4.9$$

We now have $$100 - T(x,\,t) \;=\; A\int_{xt^{-1/2}}^{0} \exp\left[-u^2/(4D)\right]du. \qquad 4.4.10$$

The error function erf(r) is defined by

$$\operatorname{erf}(r) \;=\; \frac{2}{\sqrt{\pi}}\int_0^r \exp(-s^2)\,ds, \qquad 4.4.11$$

so that equation 4.4.10 can be written

$$T(x,t) \;=\; 100 + A\sqrt{\pi D}\,\operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right) \;=\; 100\left[1 - \operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right)\right]. \qquad 4.4.12$$

This function is easy to plot provided that your computer will give you the erf function. The solutions are shown in figures IV.4 and 5.

Figure IV.4

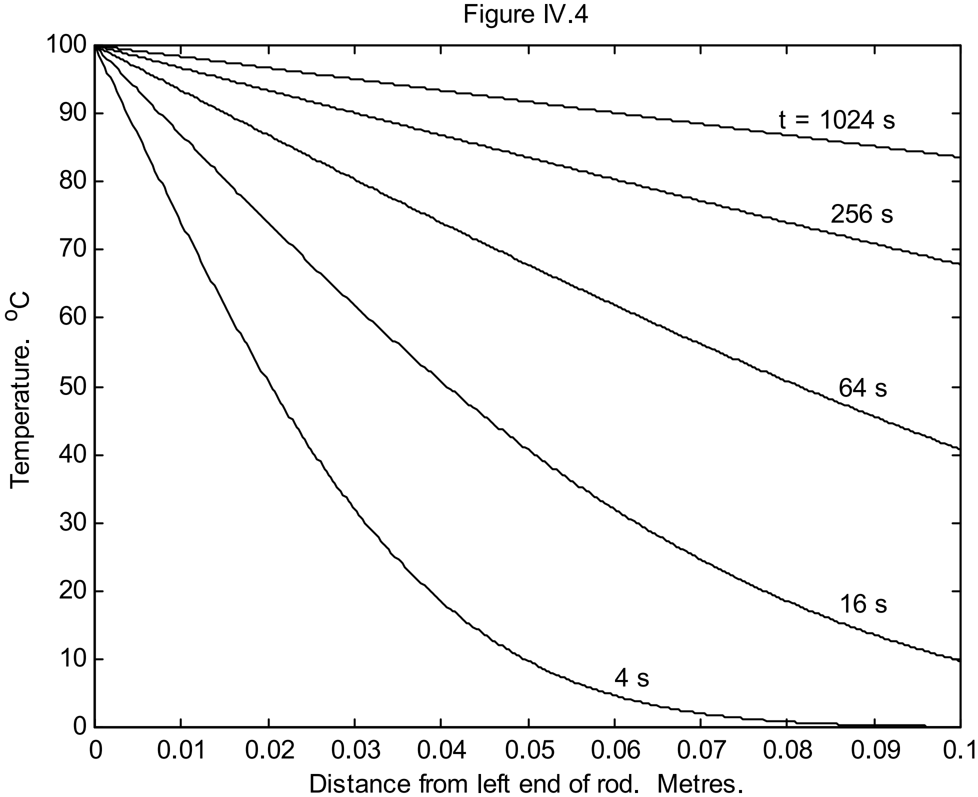
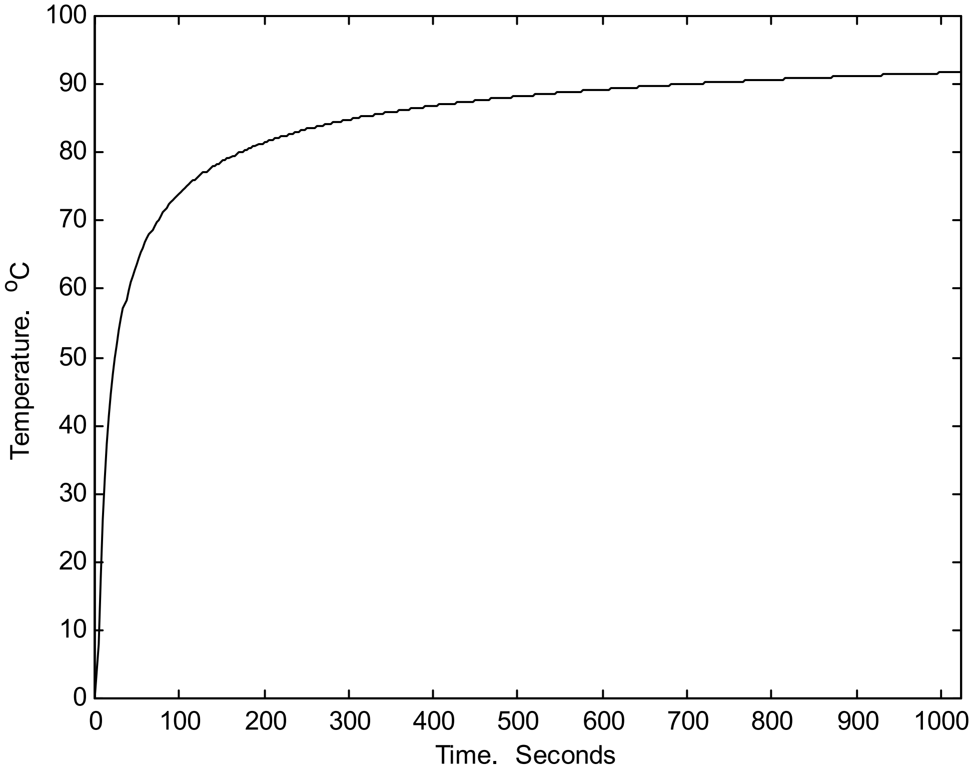


Figure IV.5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Temperature. °C
Time. Seconds