CHAPTER 5
COLLISIONS

5.1 *Introduction*

In this chapter on collisions, we shall have occasion to distinguish between elastic and inelastic collisions. An elastic collision is one in which there is no loss of translational kinetic energy. That is, not only must no translational kinetic energy be degraded into heat, but none of it may be converted to vibrational or rotational kinetic energy. It is well known, for example, that if a ball makes a glancing (i.e. not head-on) elastic collision with another ball of the same mass, initially stationary, then after collision the two balls will move off at right angles to reach other. But this is so only if the balls are smooth. If they are rough, after collision the balls will be spinning, so this result – and any other results that assume no loss of translational kinetic energy – will not be valid. When molecules collide, they may be set into rotational and vibrational motion, and in that case the collision will not be elastic in the sense in which we are using the term. If two atoms collide, one (or both) may be raised to an excited electronic level. Some of the translational kinetic energy has then been converted to potential energy. If the excited atom subsequently drops down to a lower level, that energy is radiated away and lost from the system. Superelastic collisions are also possible. If one atom, before collision, is in an excited electronic state, on collision it may make a radiationless downwards transition, and the potential energy released is then converted to translational kinetic energy, so the collision is superelastic. None of this is intended to mean that elastic collisions are impossible or even rare. In the case of collisions involving macroscopic bodies, such as smooth, hard billiard balls, collisions may not be 100% elastic, but they may be close to it. In the case of low-energy (low temperature) collisions between atoms, there need be no excitation to excited levels, in which case the collision will be elastic. Some subatomic particles, in particular leptons (of which the electron is the best-known example), are believed to have no internal degrees of freedom, and therefore collisions between them are necessarily elastic.

In laying out the principles involved in collisions between particles, we need not suppose that the particles actually "bang into" – i.e. touch – each other. For example most of the principles that we shall be describing apply equally to collisions between balls that "bang into" each other and to phenomena such as Rutherford scattering, in which an alpha particle is deviated from its path by a gold nucleus without actually "touching" it. Of course, if you think about it at an atomic level, when two billiard balls collide, the atoms don't actually "touch" each other; they are repelled from each other by electromagnetic forces, just as the alpha particle and the gold nucleus repelled each other in the Rutherford-Geiger-Marsden experiment.

The theory of collisions is used a great deal, of course, in the study of high-energy collisions between particles in particle physics. Bear in mind, however, that in "atom-smashing" experiments with modern huge particle accelerators, or even in relatively mild collisions such as Compton scattering of x-rays, the particles involved are moving at speeds that are not negligible compared with the speed of light, and therefore relativistic mechanics is needed for a proper analysis. In this chapter, collisions are treated entirely from a nonrelativistic point of view.

5.2 *Bouncing Balls*

When a ball is dropped to the ground, one of four things may happen:

1. It may rebound with exactly the same speed as the speed at which it hit the ground. This is an *elastic* collision.

2. It may come to a complete rest, for example if it were a ball of soft putty. I shall call this a *completely inelastic* collision.

3. It may bounce back, but with a reduced speed. For want of a better term I shall refer to this as a *somewhat inelastic* collision.

4. If there happens to be a little heap of gunpowder lying on the table where the ball hits it, it may bounce back with a faster speed than it had immediately before collision. That would be a *superelastic* collision.

The ratio $\frac{\text{speed after collision}}{\text{speed before collision}}$ is called the *coefficient of restitution*, for which I shall use the symbol e. The coefficient is 1 for an elastic collision, less than 1 for an inelastic collision, zero for a completely inelastic collision, and greater than 1 for a superelastic collision. The ratio of kinetic energy (after) to kinetic energy (before) is evidently, *in this situation*, e^2.

If a ball falls on to a table from a height h_0, it will take a time $t_0 = \sqrt{2h_0/g}$ to fall. If the collision is somewhat inelastic it will then rise to a height $h_1 = e^2 h_0$ and it will take a time et to reach height h_1. Then it will fall again, and bounce again, this time to a lesser height. And, if the coefficient of restitution remains the same, it will continue to do this for an infinite number of bounces. After a billion bounces, there is still an infinite number of bounces yet to come. The total distance travelled is

$$h = h_0 + 2h_0(e^2 + e^4 + e^6 + \ldots) \qquad 5.2.1$$

and the time taken is

$$t = t_0 + 2t_0(e + e^2 + e^3 + \ldots). \qquad 5.2.2$$

These are geometric series, and their sums are

$$h = h_0\left(\frac{1+e^2}{1-e^2}\right), \qquad 5.2.3$$

which is independent of g (i.e. of the planet on which this experiment is performed), and

$$t = t_0\left(\frac{1+e}{1-e}\right). \qquad 5.2.4$$

For example, suppose $h_0 = 1$ m, $e = 0.5$, $g = 9.8$ m s^{-2}, then the ball comes to rest in 1.36 s after having travelled 1.67 m after an infinite number of bounces. Discuss. (E.g. Does it ever stop bouncing, given that, after every bounce, there is still an infinite number yet to come; yet after 1.36 seconds it is no longer bouncing...?)

5.3 *Head-on Collision of a Moving Sphere with an Initially Stationary Sphere*

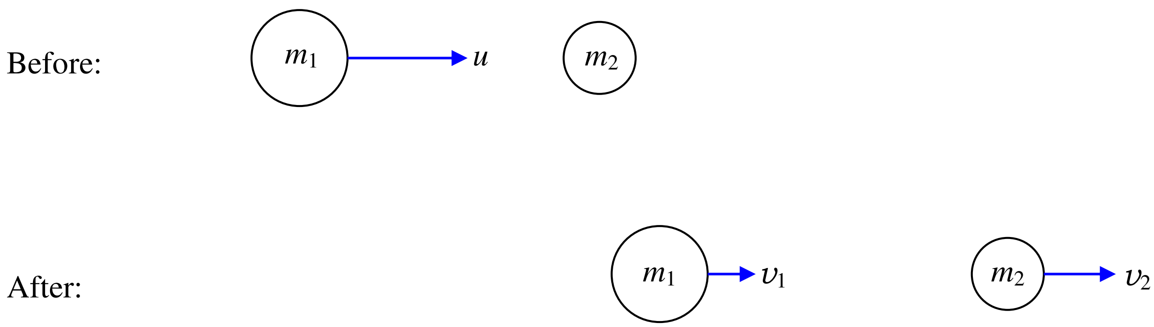


FIGURE V.1

The coefficient of restitution is

$$e = \frac{\text{relative speed of recession after collision}}{\text{relative speed of approach before collision}}\,. \qquad 5.3.1$$

We suppose that the two masses m_1 and m_2, the initial speed u, and the coefficient of restitution e are known; we wish to find v_1 and v_2. We evidently need two equations. Since there are no *external* forces on the *system*, the linear momentum of the *system* is conserved:

$$m_1 u = m_1 v_1 + m_2 v_2\,. \qquad 5.3.2$$

The second equation will be the restitution equation (equation 5.3.1):

$$v_2 - v_1 = eu\,. \qquad 5.3.3$$

These two equations can be solved to yield

$$v_1 = \left(\frac{m_1 - m_2 e}{m_1 + m_2}\right) u \qquad 5.3.4$$

and $$v_2 = \left(\frac{m_1(1+e)}{m_1 + m_2} \right) u . \qquad 5.3.5$$

The relation between the kinetic energy loss and the coefficient of restitution isn't quite as simple as in section 5.2. *Exercise*. Show that

$$\frac{\text{kinetic energy (after)}}{\text{kinetic energy (before)}} = \frac{m_1 v_1^2 + m_1 v_2^2}{m_1 u^2} = \frac{m_1 + m_2 e^2}{m_1 + m_2} . \qquad 5.3.6$$

If $m_2 = \infty$ (as in section 5.2), this becomes just e^2. If $e = 1$, it becomes unity, so all is well.

If $m_1 \ll m_2$ (Ping-pong ball collides with cannon ball), $v_1 = -u$, $v_2 = 0$.

If $m_1 = m_2$ (Ping-pong ball collides with ping-pong ball), $v_1 = 0$, $v_2 = u$.

If $m_1 \gg m_2$ (Cannon ball collides with ping-pong ball), $v_1 = u$, $v_2 = 2u$.

Example: A moving sphere has a head-on elastic collision with an initially stationary sphere. After collision the kinetic energies of the two spheres are equal. Show that the mass ratio of the two spheres is 0.1716. Which of the two spheres is the more massive? (I guarantee that your answer to this will be correct.)

5.4 *Oblique Collisions*

In figure V.2 I show two balls just before collision, and just after collision. The horizontal line is the line joining the centres – for short, the "line of centres". We suppose that we know the velocity (speed and direction) of each ball before collision, and the coefficient of restitution. The direction of motion is to be described by the angle that the velocity vector makes with the line of centres. We want to find the velocities (speed and direction) of each ball after collision. That is, we want to find four quantities, and therefore we need four equations. These equations are as follows.

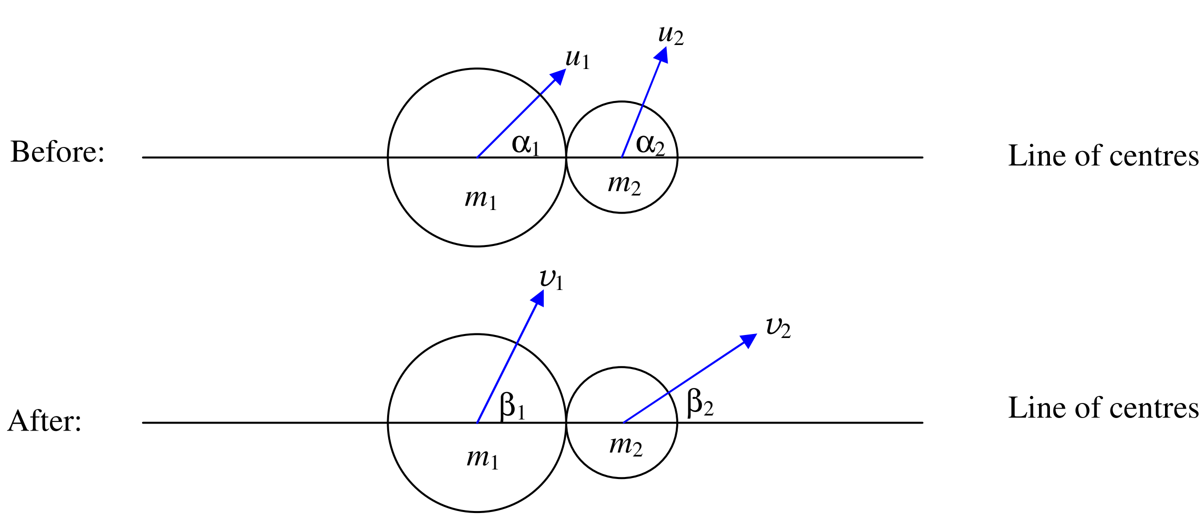


FIGURE V.2

There are no external forces on the system along the line of centres. Therefore the component of momentum of the system along the line of centres is conserved:

$$m_1 v_1 \cos\beta_1 + m_2 v_2 \cos\beta_2 = m_1 u_1 \cos\alpha_1 + m_2 u_2 \cos\alpha_2. \qquad 5.4.1$$

If we assume that the balls are smooth -.i.e. that there are no forces perpendicular to the line of centres and the balls are not set into rotation, then the component of the momentum of each ball separately perpendicular to the line of centres is conserved:

$$v_1 \sin\beta_1 = u_1 \sin\alpha_1 \qquad 5.4.2$$

and

$$v_2 \sin\beta_2 = u_2 \sin\alpha_2 . \qquad 5.4.3$$

The last of the four equations is the restitution equation

$$e = \frac{\text{relative speed of recession along the line of centres after collision}}{\text{relative speed of approach along the line of centres before collision}} .$$

That is,

$$v_2 \cos\beta_2 - v_1 \cos\beta_1 = e(u_1 \cos\alpha_1 - u_2 \cos\alpha_2). \qquad 5.4.4$$

Example. Suppose $m_1 = 3$ kg, $m_2 = 2$ kg, $u_1 = 40$ m s^{-1} $u_2 = 15$ m s^{-1}
$\alpha_1 = 10^\circ$, $\alpha_2 = 70^\circ$, $e = 0.8$

Find v_1, v_2, β_1, β_2.

Answers: $v_1 = 16.28$ m s^{-1} $v_2 = 44.43$ m s^{-1}
$\beta_1 = 25^\circ\ 15'$ $\beta_2 = 18^\circ\ 30'$

Example. Suppose $m_1 = 2$ kg, $m_2 = 3$ kg, $u_1 = 12$ m s^{-1} $u_2 = 15$ m s^{-1}
$\alpha_1 = 20^\circ$, $\alpha_2 = 50^\circ$, $\beta_2 = 47^\circ$

Find v_1, v_2, β_1, e.

Answers: $v_1 = 10.50$ m s^{-1} $v_2 = 15.71$ m s^{-1}
$\beta_1 = 23^\circ\ 00'$ $e = 0.6418$

Problem. If $u_2 = 0$, and if $e = 1$ and if $m_1 = m_2$, show that $\beta_1 = 90^\circ$ and $\beta_2 = 0^\circ$.

5.5 *Oblique (Glancing) Elastic Collisions, Alternative Treatment*

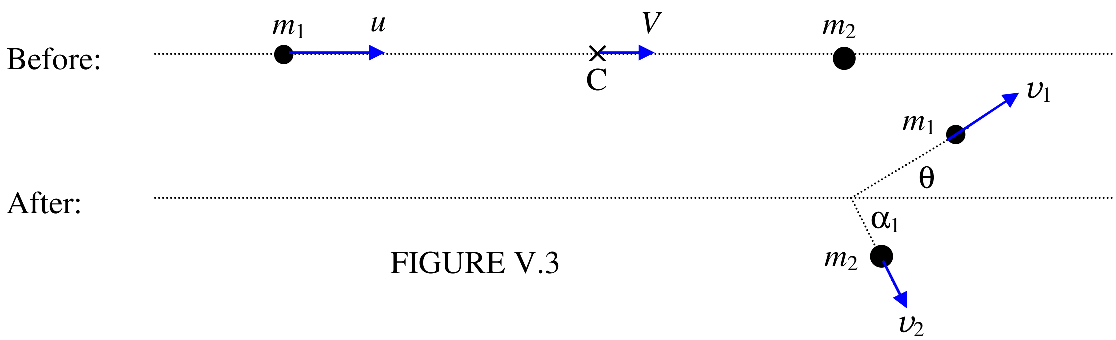


FIGURE V.3

In figure V.3, unlike figure V.2, the horizontal line is not intended to represent the line of centres. Rather, it is the direction of the initial velocity of m_1, and m_2 is initially at rest. The second mass m_2 is slightly off the line of the velocity of m_1. I am assuming that the collision is elastic, so that $e = 1$. In the "before" part of the figure, I have indicated, as well as the two masses, the position and velocity **V** of the centre of mass C. The velocity of C remains constant, because there are no external forces on the system. I have not drawn C in the "after" part of the figure, because it would get a little in the way. Think about where it is.

Figure V.3 shows the situation in "laboratory space". (Later, we'll look at the situation referred to a reference frame in which C is at rest – "centre of mass space".) The angle θ is the angle through which m_1 has been scattered (the "scattering angle"). I have indicated in the figure how it is related to the α_1 and β_1 of section 5.3. Note that m_2 (initially stationary) scoots off along the line of centres.

The following two equations express the constancy of linear momentum of the system.

$$(m_1+m_2)V = m_1 u = m_1 \upsilon_1 \cos\theta + m_2 \upsilon_2 \cos\alpha_1. \qquad 5.5.1$$

I'm going to draw, in figure V.4, the situation "close-up", so that you can see the geometry more clearly. Note that the distance b is called the *impact parameter*. It is the distance by which the two centres would have missed each other had the first particle not been scattered.

In figure V.5, I draw the situation in centre of mass space, in which the centre of mass C is stationary. In this reference frame, I just have to subtract **V** from all the velocities. Note that in centre of mass space the *speeds* of the particles are unaltered by the collision. In centre of mass space, m_1 is scattered through an angle θ', and I am going to find a relation between θ' , θ and the mass ratio m_2/m_1.

I shall start with the profound statement that

$$\tan\theta = \frac{\upsilon_1 \sin\theta}{\upsilon_1 \cos\theta}. \qquad 5.5.2$$

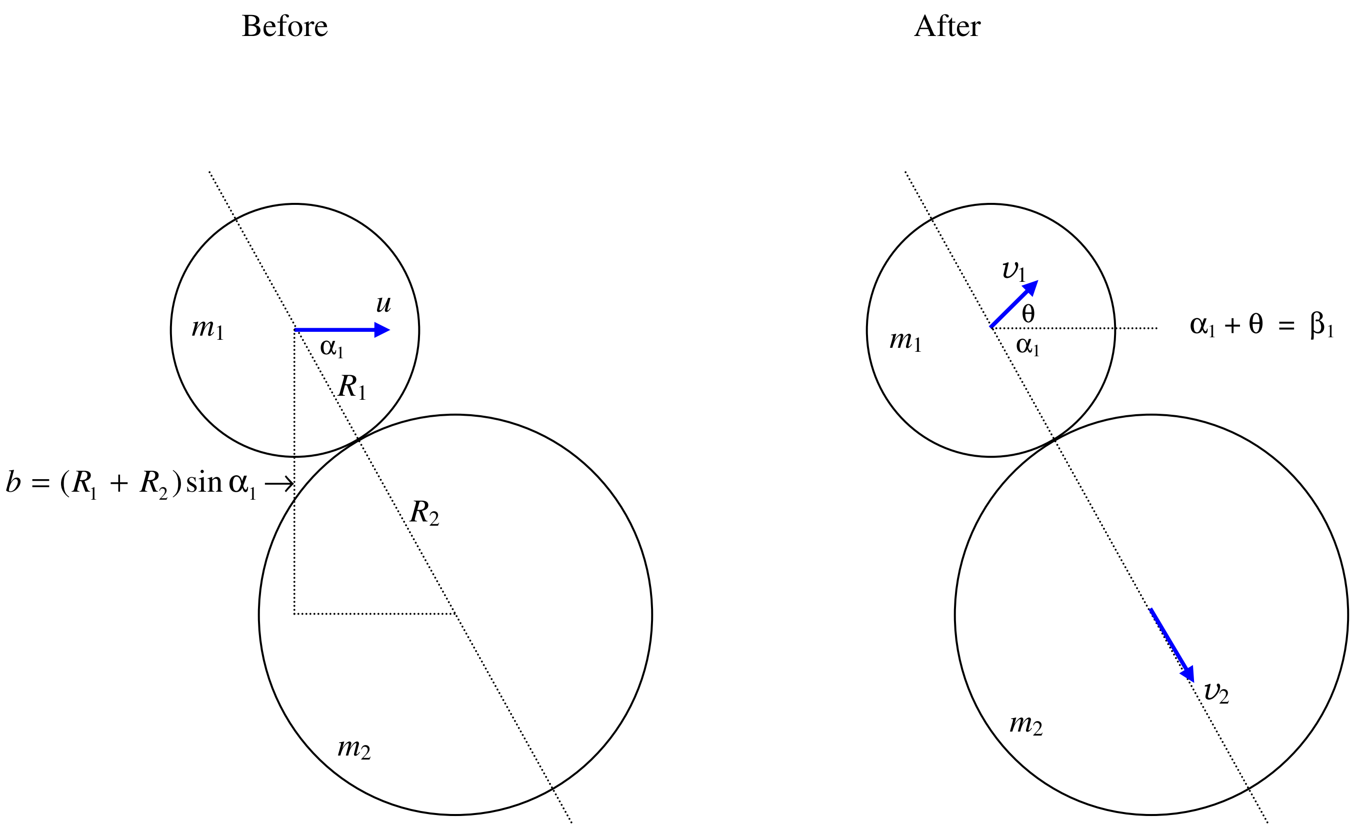


FIGURE V.4

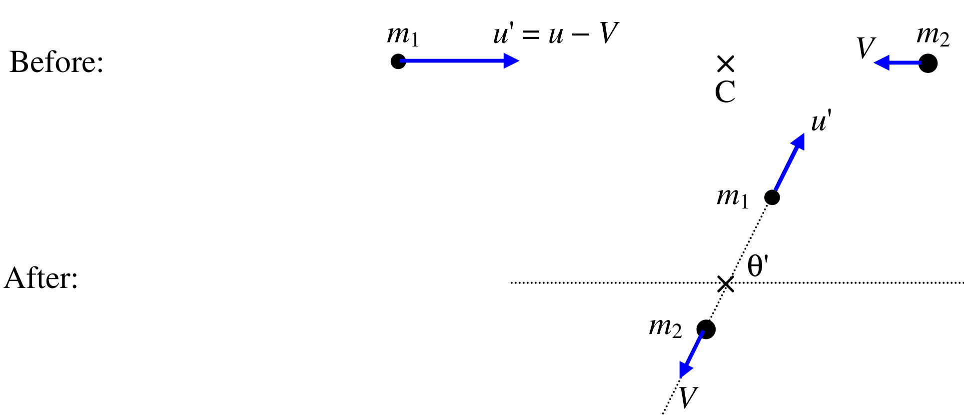


FIGURE V.5

Now $\upsilon_1 \sin\theta$ is the y-component of the final velocity of m_1 in laboratory space. The y-component of the final velocity of m_1 in centre of mass space is $u'\sin\theta'$, and these two are equal, since the y-component of the motion is unaffected by the change of reference frame. Therefore

$$\tan\theta = \frac{u'\sin\theta'}{\upsilon_1\cos\theta}. \qquad 5.5.3$$

Therefore $$\upsilon_1\sin\theta = u'\sin\theta'. \qquad 5.5.4$$

The x-components of the "before" and "after" velocities of m_1 are related by

$$\upsilon_1\cos\theta = u'\cos\theta' + V. \qquad 5.5.6$$

Substitute equations 5.5.4 and 5.5.6 into equation 5.5.2 to obtain

$$\tan\theta = \frac{\sin\theta'}{\cos\theta' + V/u'}. \qquad 5.5.7$$

But $$(m_1+m_2)V = m_1 u = m_1(u'+V), \qquad 5.5.8$$

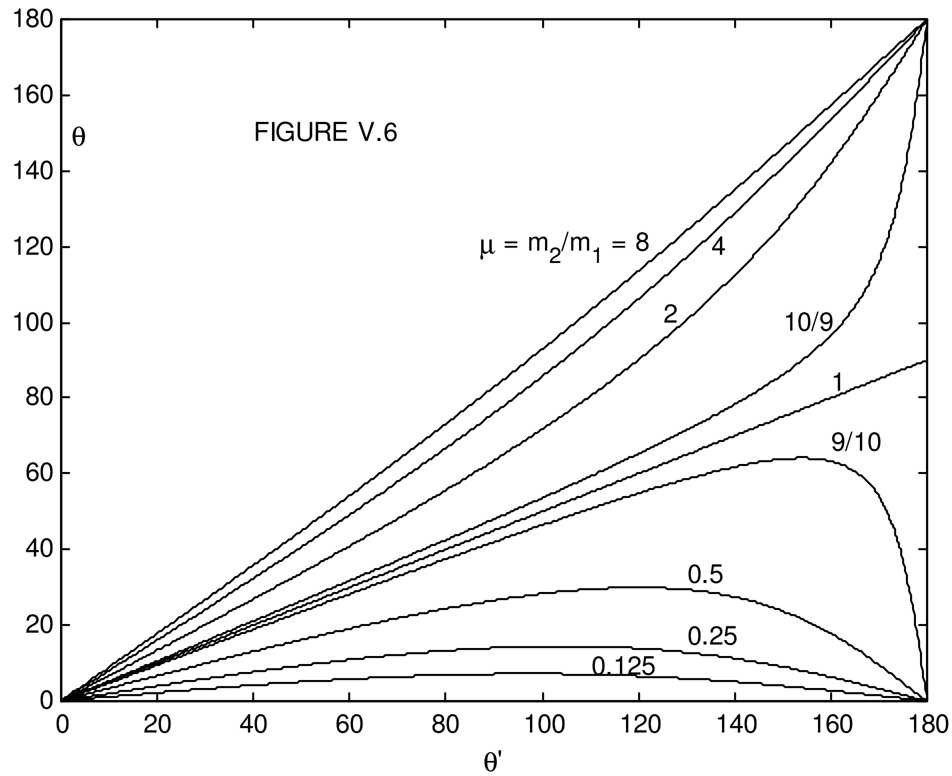
from which $$\frac{V}{u'} = \frac{m_1}{m_2}. \qquad 5.5.9$$

On substituting this into equation 5.5.7, we obtain the relation we sought:

$$\tan\theta = \frac{\sin\theta'}{\cos\theta' + m_1/m_2}. \qquad 5.5.10$$

This relation is illustrated in figure V.6 for several mass ratios.

Let us try to interpret the figure. For $m_2 > m_1$, any scattering angle, forward or backward, is possible, but for $m_2 < m_1$, backward scattering is not possible, and forward scattering is possible only up to a maximum. This is only to be expected. Thus for an impact parameter of zero or of $R_1 + R_2$, and $m_2 < m_1$, the scattering angle θ must be zero, and therefore for intermediate impact parameters it must go through a maximum. This would be clearer if we could plot the scattering angle versus the impact parameter, and indeed that is something that we shall try to do. In the meantime it is easy to show, by differentiation of equation 5.5.10 (do it!), that the maximum scattering angle is $\sin^{-1}\mu$, where $\mu = m_2/m_1$. That is, if the scattered particle is very massive compared with the scattering particle, the maximum scattering angle is small – just to be expected.



I want to do two things now - <u>one</u>, to calculate the scattering angle θ as a function of impact parameter, and <u>two</u>, to calculate υ_1/u as a function of scattering angle. I'm going to start with equations 5.4.1,2 and 4, except for the following. I'll assume $e = 1$ (elastic collision), and $u_2 = 0$ (m_2 is initially stationary), and $\beta_2 = 0$ (since m_2 is initially stationary, it must move along the line of centres after collision). Since I want to try to calculate the scattering angle, I'll write $\theta + \alpha_1$ for β_1 (see figure V.4). I'm also going to write r_1 , r_2 and μ for the dimensionless ratios υ_1/u, υ_2/u and m_2/m_1 respectively. With those small changes, equations 5.4.1,2 and 4 become

$$r_1 \cos(\theta + \alpha_1) + \mu r_2 = \cos\alpha_1, \qquad 5.5.11$$

$$r_1 \sin(\theta + \alpha_1) = \sin\alpha_1, \qquad 5.5.12$$

$$r_2 - r_1 \cos(\theta + \alpha_1) = \cos\alpha_1. \qquad 5.5.13$$

Eliminate r_2 from equations 5.5.11 and 5.5.13 to obtain

$$r_1 \cos(\theta + \alpha_1) = M \cos\alpha_1, \qquad 5.5.14$$

where
$$M = \frac{1-\mu}{1+\mu} = \frac{m_1 - m_2}{m_1 + m_2}. \qquad 5.5.15$$

If we now eliminate α_1 from equations 5.5.12 and 14, we obtain the relation between v_1/u and the scattering angle, which was the second of our two aims above. The elimination is easily done as follows. Expand sin and cos of $\theta + \alpha_1$ in the two equations, divide both sides of each equation by cos α_1 and eliminate tan α_1 between the two equations. The result is

$$r_1^2 - r_1(1+M)\cos\theta + M = 0. \qquad 5.5.16$$

We'll have a look at this equation in a moment, but in the meantime, instead of eliminating α_1 from equations 5.5.12 and 14, let's eliminate r_1. This will give us a relation between the scattering angle θ and α_1, and, since α_1 is closely relation to the impact parameter (see figure V.4) this will achieve the first of our aims, namely to find the scattering angle as a function of the impact parameter. If you do the algebra, you should find that the relation between θ and α_1 is

$$t = \frac{a(1-M)}{a^2 + M}, \qquad 5.5.17$$

where $$t = \tan\theta \quad \text{and} \quad a = \tan\alpha_1. \qquad 5.5.17\text{a,b}$$

Now let $$b' = \frac{b}{R_1 + R_2} \qquad 5.5.18$$

and from figure V.5 we see that $$b' = \sin\alpha_1. \qquad 5.5.19$$

On elimination of α_1 from equations 5.5.17 and 5.5.19, we obtain the required relation between scattering angle θ and (dimensionless) impact parameter b':

$$\tan\theta = \frac{2\mu b'\sqrt{1-b'^2}}{1-\mu+2\mu b'^2}. \qquad 5.5.20$$

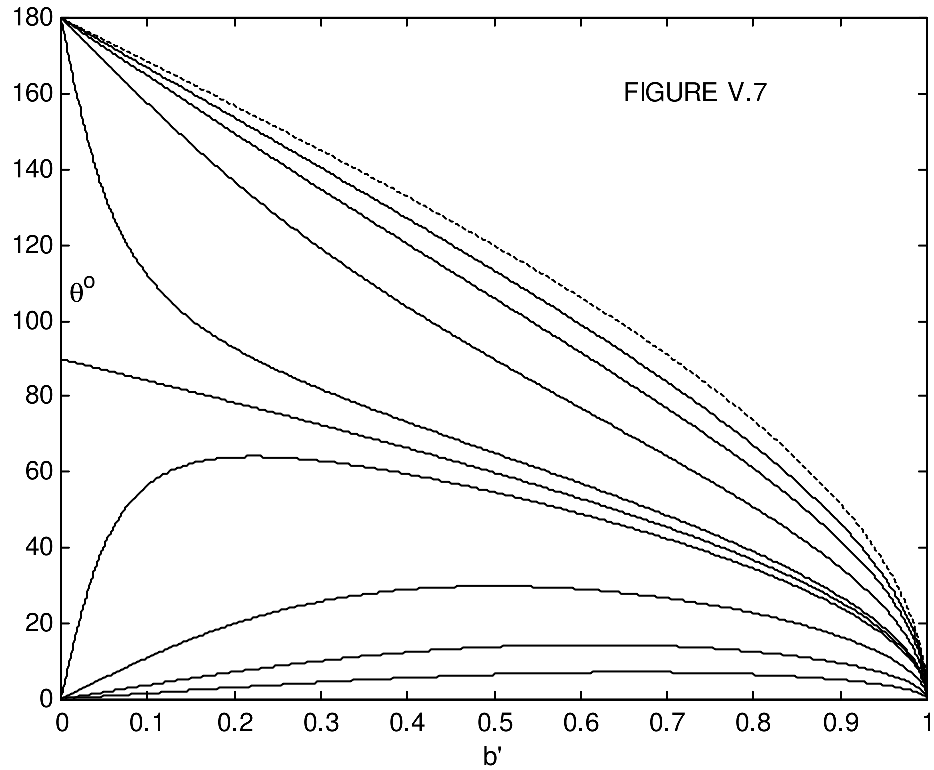
This relation is shown in figure V.7. The values of the mass ratio μ $(= m_2/m_1)$ are (from the lowest up) $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{9}{10}$, 1, $\frac{10}{9}$, 2, 4, 8 and (dashed) ∞. This figure is perhaps slightly easier to interpret than figure V.6. One can see that for $\mu > 1$, any scattering angle is possible, but for $\mu < 1$, the scattering angle has a maximum possible value, less than 90°, and the scattering angle is zero for $b' = 0$ or 1.

Exercise. We saw, by differentiation of equation 5.5.10, that the maximum scattering angle was $\sin^{-1}\mu$. Now show the same thing by differentiation of equation 5.5.20. (This is not so easy, is it?) Show that the scattering angle is greatest for an impact parameter of

$$b' = \sqrt{\frac{1-\mu}{2}}. \qquad 5.5.21$$

You will notice that, for $b' = 0$ (head-on collision) the scattering angle changes abruptly from 0 to 180° as the mass ratio changes from less than 1 to more than 1. No problem there. But if the mass ratio is *exactly* 1 (not the tiniest bit less or the tiniest bit more) the scattering angle is apparently 90°. This may cause some puzzlement until it is realized that for a head-on collision with $\mu = 1$ the first sphere comes to a dead halt.

The case of $\mu = \infty$ (second sphere immovable) is of some interest. It is easy in that case to calculate how the scattering angle varies with impact parameter for an elastic collision, merely by requiring the scattered sphere to obey the law of reflection, and without any reference to equation 5.5.20.



Easy Exercise. Without any reference to equation 5.5.20, show that, if the second ball is immovable, the scattering angle is related to the impact parameter by

$$\theta = 180^\circ - 2\sin^{-1} b'. \qquad 5.5.22$$

Not-so-easy exercise. Show that, in the limit as $\mu \to \infty$, equation 5.5.20 approaches equation 5.5.22.

In any case, the limiting case as the second sphere becomes immovable is shown as a dashed curve in figure V.7.

Exercise of Intermediate Difficulty. The mass ratio m_2/m_1 is 0.9, and the scattering angle is 50°. What was the impact parameter?

Answers b' = 0.07270 or 0.58540.

We have now dealt with the *direction of motion* of m_1 after scattering as a function of impact parameter. We should now look at the *speed* of m_1 after collision, and this takes us back to equation 5.5.16, which gives is the speed $(r_1 = v_1/u)$ as a function of scattering angle θ. It is quadratic in r_1, so, for a given scattering angle there are two possible speeds – which is not surprising, because a given scattering angle can arise from two different impact parameters, as we have just found out. We can conveniently show the relation between r_1 and θ simply by plotting the equation in polar coordinates. I'll re-write the equation here for easy reference:

$$r_1^2 - r_1(1+M)\cos\theta + M = 0. \qquad 5.5.16$$

Here, $M = \dfrac{1-\mu}{1+\mu} = \dfrac{m_1 - m_2}{m_1 + m_2}$, but I want to write the equation in terms of the *mass fractions*

$$q_1 = \frac{m_1}{m_1+m_2} = \frac{1}{1+\mu} \quad \text{and} \quad q_2 = \frac{m_2}{m_1+m_2} = \frac{\mu}{1+\mu}\,. \qquad 5.5.23\text{a,b}$$

If you work at this for a short while, you will find that equation 5.5.16 becomes

$$r_1^2 + q_1^2 - 2r_1q_1\cos\theta = q_2^2, \qquad 5.5.24$$

and one is then overcome with an overwhelming desire to draw a triangle:

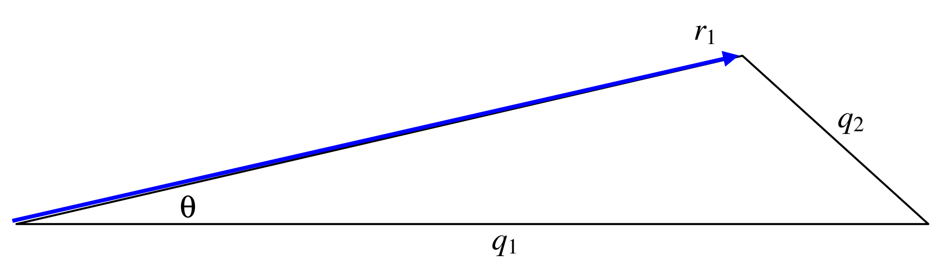


FIGURE V.8

For a given mass ratio, the locus of r_1 (the speed) versus θ (the scattering angle) is such that q_1 and q_2 are constant – in other words, it is a circle:

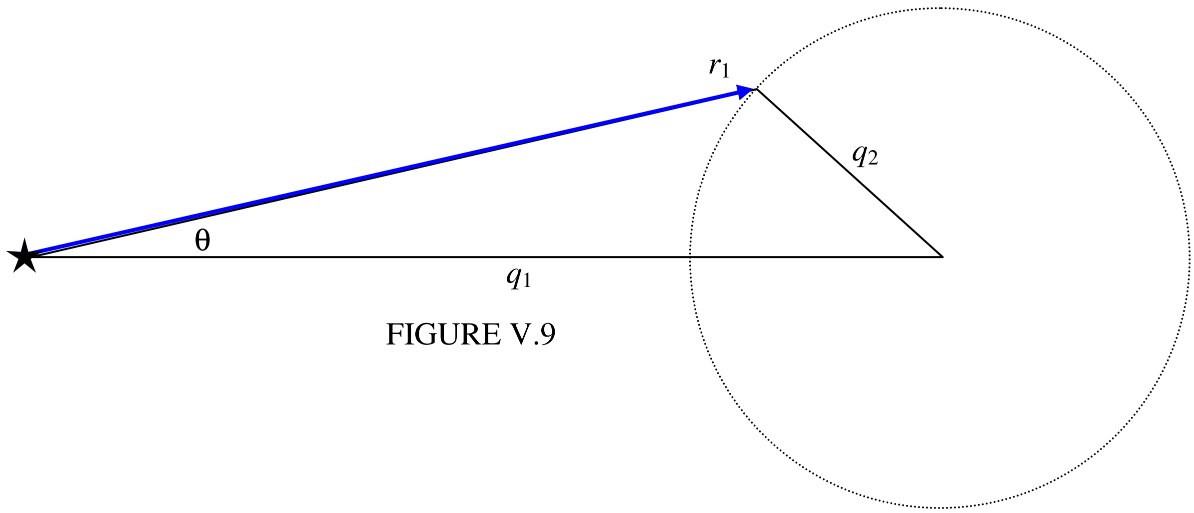


FIGURE V.9

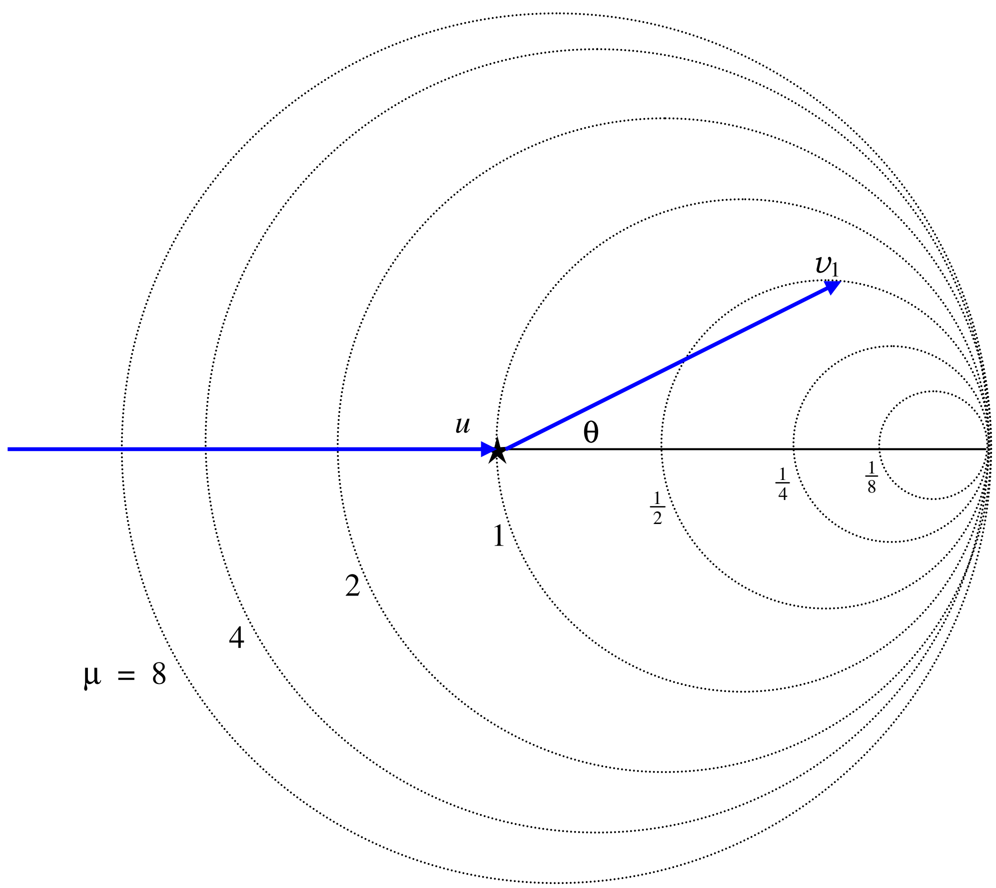


FIGURE V.10

One can imagine that the first particle comes in from the left at speed u and the collision takes place at the asterisk, and, after collision, it is moving at a speed r_1 times u in a direction θ, the magnitude of its velocity vector being determined by where the vector intersects the circle (in two possible places) given by equation 5.5.24. The maximum scattering angle corresponds to a velocity vector that is tangent to the circle. If the asterisk is the pole (origin) of the polar coordinates, the centre of the circle is at a distance q_1 from the pole, and its radius is q_2. Figure V.10 shows the circles corresponding to several mass ratios. The figure graphically illustrates the relation between u, υ_1 , θ and μ. You can see, for example, that if $\mu > 1$, scattering through any angle is possible, and the relation between υ_1 and θ is unique; but if $\mu < 1$, only forward scattering is possible, up to a maximum θ, and, for a given θ, there are two solutions for υ_1.

This deals with what happens to the sphere m_1. We can now turn our attention to m_2. Starting from equations 5.5.11, 12 and 13, we are going to want to eliminate r_1 and θ - indeed anything that pertains to the sphere m_1.

If you refer to figure V.4 you will see that, after collision, m_2 scoots off at an angle α_1 to the original direction of motion of m_1. Therefore I think it is of interest to find a relation between $r_2\ (=\upsilon_2/u)$ and α_1. If we succeed in doing this, it means that we can also find a relation, if we want it, between r_2 and the impact parameter, since $b' = \sin\alpha_1$. It is easy to eliminate r_1 from equations 5.5.12 and 13, and then you can get tan $(\theta + \alpha_1)$ from equation 5.5.14, and hence get the required relation:

$$r_2 = \frac{2\cos\alpha_1}{1+\mu}. \qquad 5.5.25$$

I'll draw this relation as a polar graph, r_2 versus α_1, in figure V.11. I'll leave the reader to work out and draw the relation between r_2 and b' if he or she wishes. Equation V.11 is the polar equation to a circle of radius $1/(1 + \mu)$.

Example. Suppose the mass ratio $\mu = m_2/m_1 = 0.5$ and the scattering angle is $\theta = 20^\circ$. Equation 5.5.16 or figure V.10 will show that $r_1 = 0.8696$ or 0.3833. Equation 5.5.17 will show that $\alpha_1 = 58^\circ.4$ or $11^\circ.6$. And equation 5.5.25 or figure V.11 will show that $r_2 = 0.6983$ or 1.306. I'll leave it to the reader to determine which alternative values of r_1 , r_2 and α_1 go together.

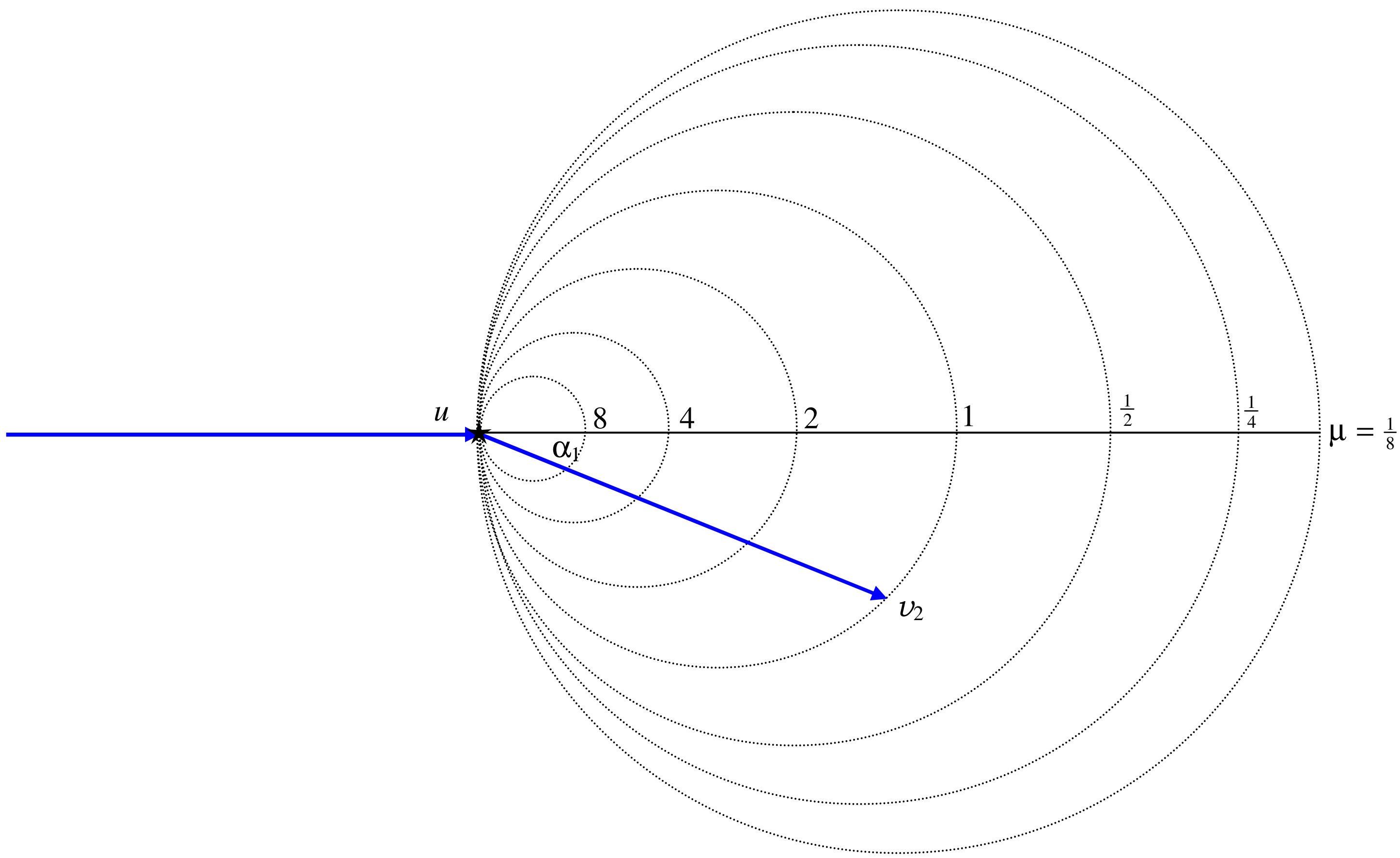


FIGURE V.11

5.6 *Two Colliding Rectangular Blocks*

I am indebted to Raunak Kumar for suggesting this problem.

Two rectangular blocks are sliding towards each other on a smooth horizontal table as shown in figure V.12. Describe the motion after collision.

That means: If they each have speed v_0 before collision, calculate the speed v and the angular speed ω after collision as a function of impact parameter d. Also calculate the ratio (rotational kinetic energy)/(total kinetic energy) as a function of d.

We refer all motions to the centre of mass, C, of the system. In the absence of any external forces, the position of the centre of mass remains fixed, before, at, and after the moment of collision.

Dimensions of each block $=$ $2a \times 2b \times 2c$

The rotational inertia of each block about its centre of mass is $I = \frac{1}{3}m(a^2 + b^2)$.

Before collision

Linear momentum of the system = 0.
Angular momentum of the system = $m\upsilon_0 d$ clockwise.

Kinetic energy of the system = $m\upsilon_0^2$.

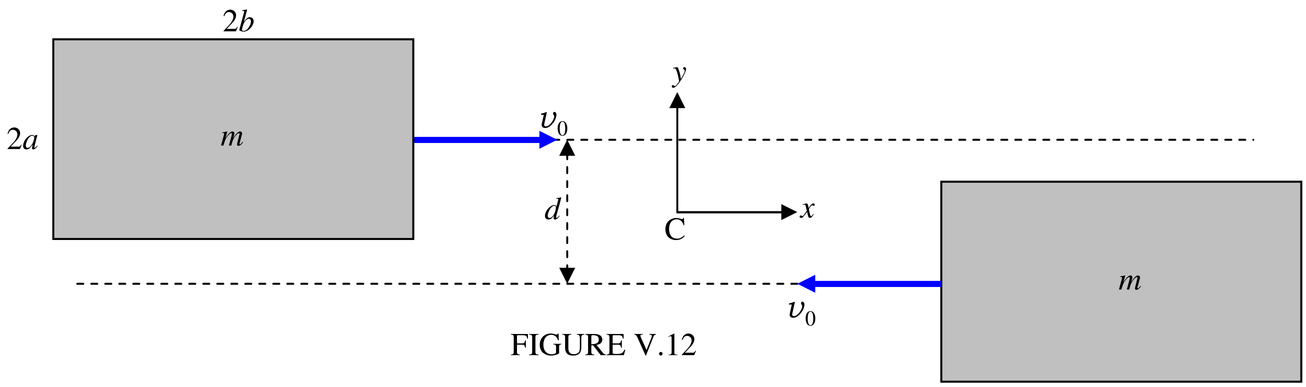


FIGURE V.12

I deal with two extreme cases (and I briefly mention an intermediate case). First, I assume that the blocks are rigid; they cannot vibrate or sustain any internal motions. The collision is completely elastic, and there is no loss of kinetic energy. Second, I assume that the collision is completely inelastic, and they stick together upon impact.

First: the completely elastic case.

After collision

Linear momentum of the system = 0.
Angular momentum of the system = $2I\omega - m\upsilon d$ clockwise.

Kinetic energy of the system = $m\upsilon^2 + I\omega^2$.

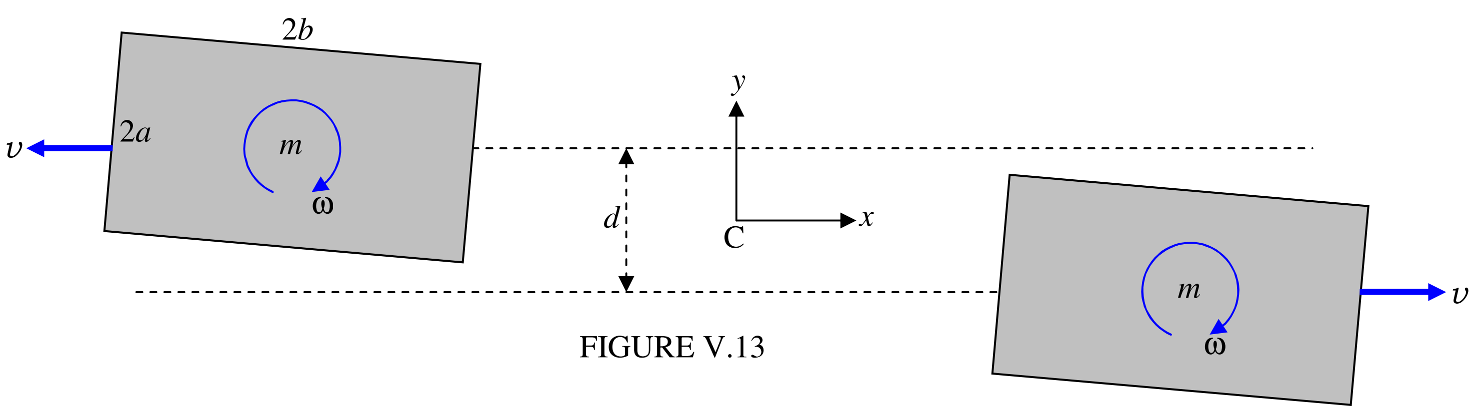


FIGURE V.13

If there are no external forces, the linear momentum of the system is conserved. It is zero before and after collision.

If there are no external torques, the angular momentum of the system is conserved, and if the blocks are rigid, kinetic energy is conserved.

Thus $$mv_0 d = 2I\omega - mvd \tag{5.6.1}$$

and $$mv_0^2 = mv^2 + I\omega^2 \;. \tag{5.6.2}$$

[Should the blocks not be rigid, the question would have to specify the fraction f of energy converted into heat. The right hand side of equation (5.6.2) would then have to be multiplied by $1 - f$.]

On elimination of ω from equations (5.6.1) and (5.6.2) we obtain

$$v = \frac{4I - md^2}{4I + md^2} v_0 \,. \tag{5.6.3}$$

On elimination of v from equations (5.6.1) and (5.6.2) we obtain

$$\omega = \frac{4md}{4I + md^2} v_0 \,. \tag{5.6.4}$$

The ratio of rotational kinetic energy after collision to total kinetic energy is

$$\frac{K_{\text{rot}}}{K_{\text{total}}} = \frac{I\omega^2}{I\omega^2 + mv^2} \tag{5.6.5}$$

Numerical example:

Suppose: $a = 4$ cm
$b = 6$ cm
$c = 1$ cm
density $\rho = 9$ g cm^{-3} (copper)

Then $m = 8abc\rho = 1728$ g
and $I = \frac{1}{3} m(a^2 + b^2) = 2.9952 \times 10^4$ g cm^2

Suppose $v_0 = 50$ cm s^{-1}

The graphs below show υ, ω and $\frac{K_{\text{rot}}}{K_{\text{total}}}$ as a function of impact parameter d.

FIGURE V.14

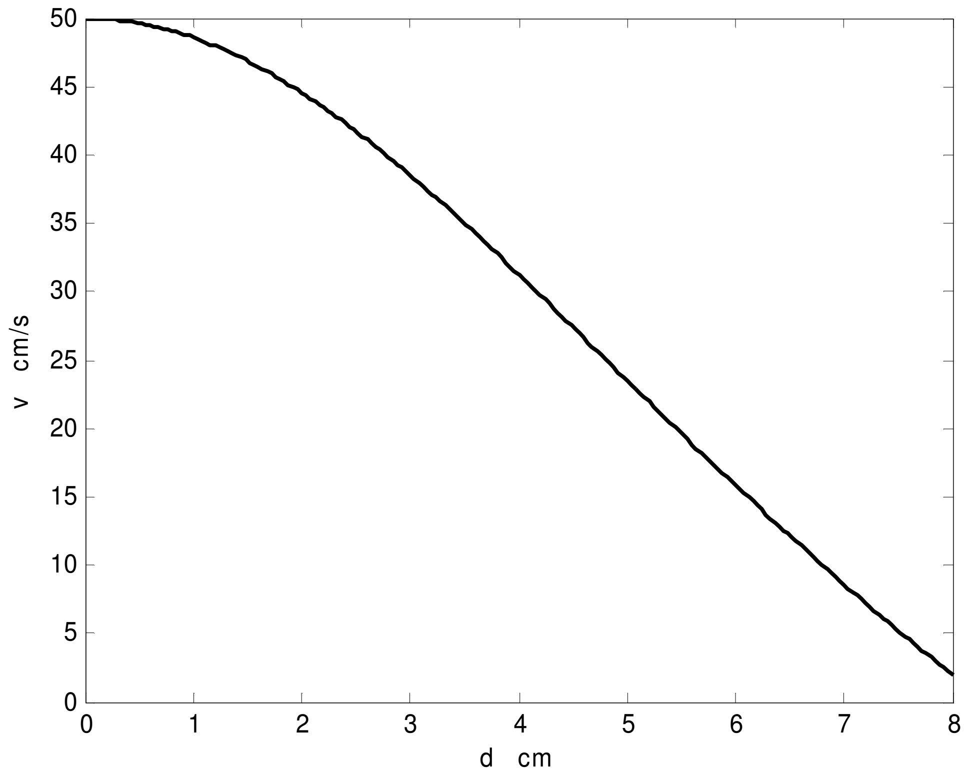


FIGURE V.15

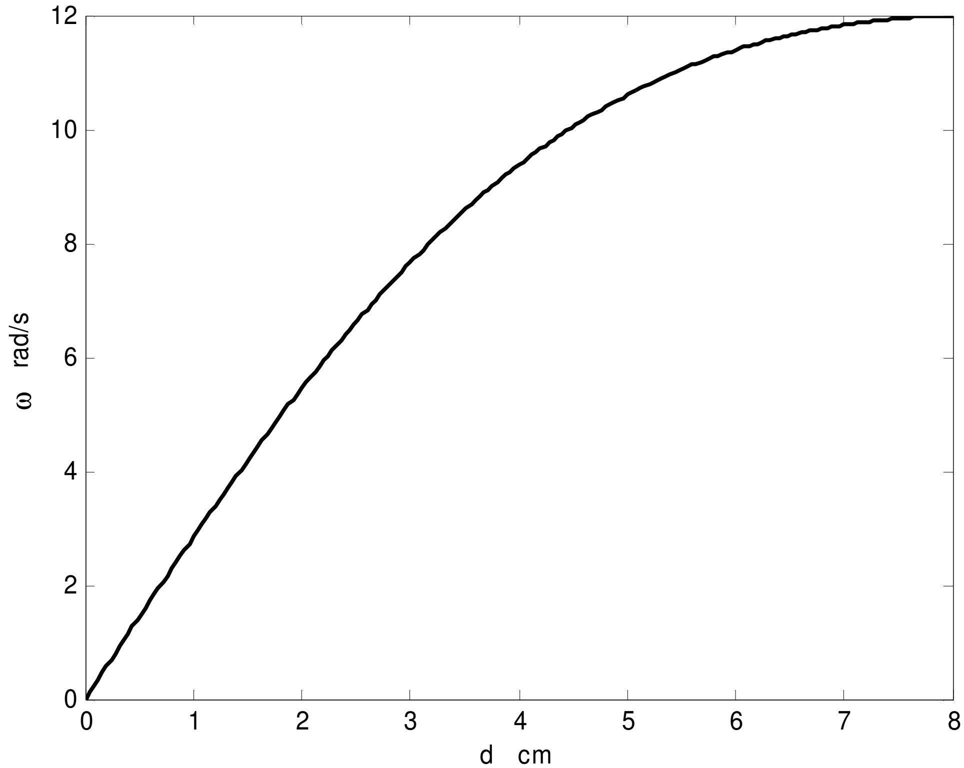
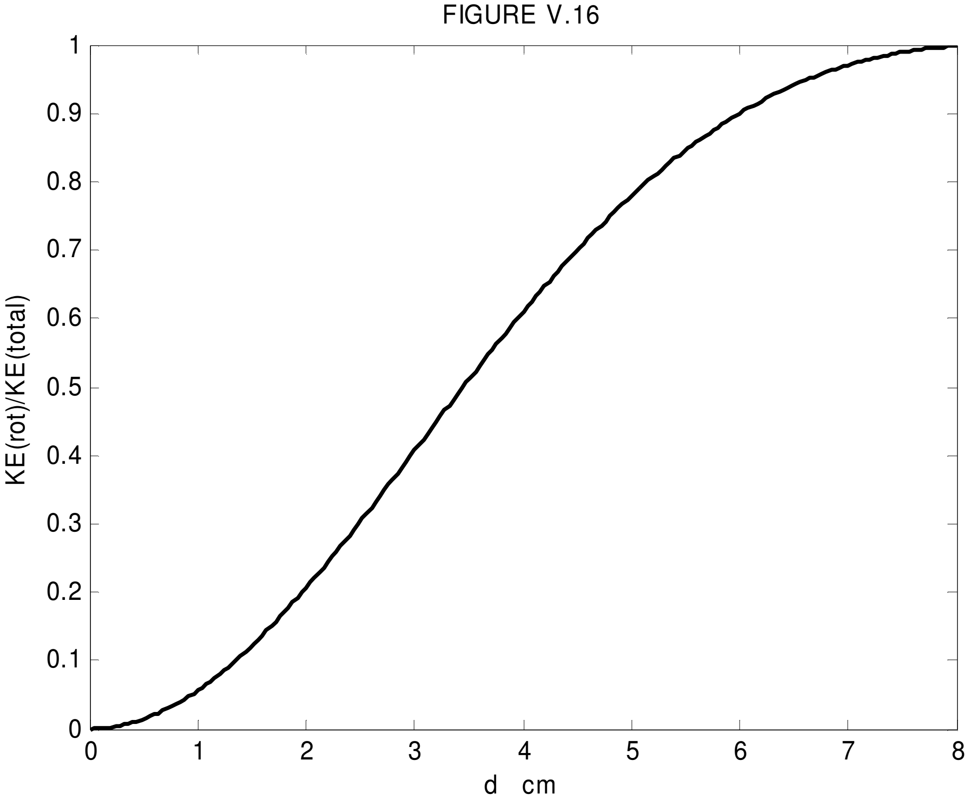


FIGURE V.16



Second: the completely inelastic case.

After collision

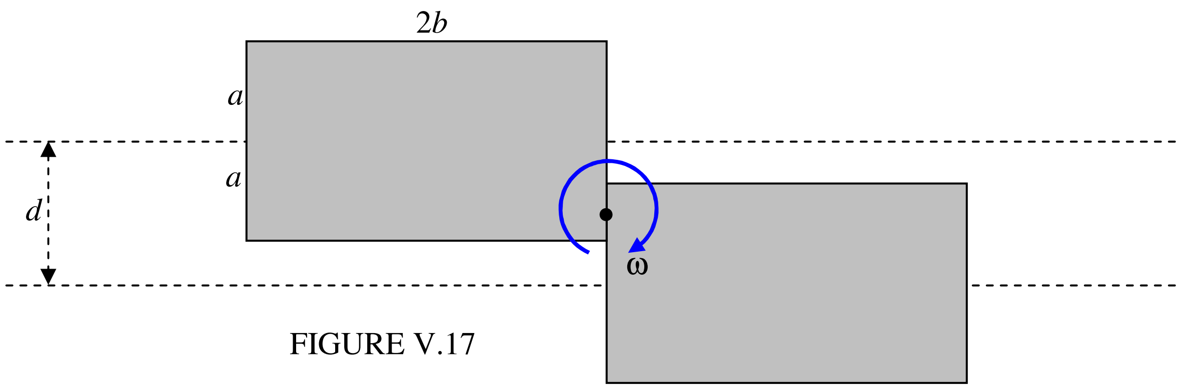


FIGURE V.17

The moment of inertia of a single block about its centre of mass is $I = \frac{1}{3}m(a^2 + b^2)$. But now we need to know the moment of inertia I_2 of the combined adhering blocks about the centre of mass of the system. I make it

$$I_2 = \frac{1}{2}\left(\frac{4a^2 + 16b^2 + 3d^2}{a^2 + b^2}\right)I \qquad 5.6.6$$

(Let me know - jtatum at uvic dot ca - if you think I got it wrong.)

Before collision the linear momentum of the system referred to the centre of mass of the system is zero. In the absence of any external forces, the linear momentum of the system is conserved; thus the linear momentum of the two adhering blocks is zero.

Before collision the angular momentum of the system referred to the centre of mass of the system is mv_0d clockwise. After collision, the angular momentum is $I_2\omega$. In the absence of any external torques, angular momentum is conserved, and therefore

$$\omega = \frac{mv_0d}{I_2}. \qquad 5.6.7$$

That is, ω increases linearly with impact parameter d. For $d = 0$, the angular speed after collision is, unsurprisingly, zero.

Before collision, all the kinetic energy is translational kinetic energy. After collision, all the kinetic energy is rotational kinetic energy, except for a head-on collision, where $d = 0$. In this case all kinetic energy is lost upon collision, and converted into heat.

The ratio (rotational kinetic energy after collision) ÷ (translational kinetic energy before collision) is

$$\frac{\frac{1}{2}I_2\omega^2}{mv_0^2} = \frac{md^2}{2I_2}. \qquad 5.6.8$$

For a glancing collision $(d = 2a)$, this becomes $\frac{2ma^2}{I_2}$.

Using the same numerical data as we used for the completely elastic case, I show below the ratio (rotational kinetic energy after collision) ÷ (translational kinetic energy before collision) as a function of impact parameter d.

FIGURE V.18