CHAPTER 20
MISCELLANEA

20.1 *Introduction*

This chapter is a miscellany of diverse and unrelated topics – namely surface tension, shear modulus and viscosity – discussed only for the purpose of presenting a few more examples of elementary problems in mechanics. It is not intended in any way to substitute for a comprehensive course in any of the vast and interesting fields of surface chemistry, elasticity or hydrodynamics. All of these subjects have a huge and specialized literature, each worthy of a full-length course, and I am not remotely competent to offer one. Nevertheless, the few simple problems chosen in this chapter are suitable for a bit more practice in classical mechanics.

20.2 *Surface Tension*

The cause of surface tension is often explained roughly as follows. Molecules within a liquid are subject to intermolecular forces whose exact nature and origin need not concern us other than to say that they are principally van der Waals forces and they hold the liquid together and prevent it from evaporating.   A molecule deep within the liquid is surrounded in all directions by other molecules, and so the net force on it averages zero. But a molecule on the surface experiences forces from beneath the surface, and consequently it tends to get dragged beneath the surface. This results in as few molecules as possible remaining on the surface; i.e. it results in the surface contracting to as small an area as possible consistent with whatever other geometrical constraints may exist. That is, the surface appears to be in a state of tension causing it to contract to the least possible area.

This tension can be described qualitatively thus.   In figure XX.1, the dashed line is an imaginary
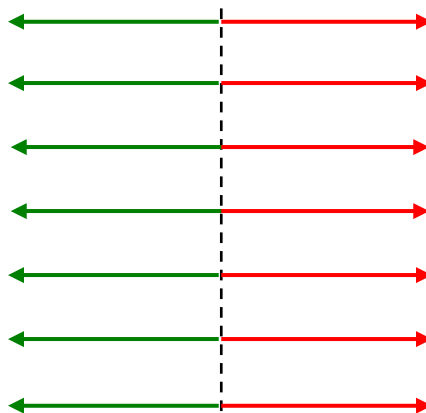
FIGURE XX.1

line drawn in the surface of a liquid. The liquid to the left of the line is being pulled to the right as indicated by the red arrows; the liquid to the right of the line is being pulled equally to the left as indicated by the green arrows. The force per unit length perpendicular to a line drawn in the

surface of the liquid is the *surface tension*. Its SI unit is newtons per metre, and its CGS unit is dynes per centimetre. The dimensions are MT^{-2}.

I have seen various symbols, such as T, S and γ used for surface tension. The first two of these symbols are already heavily worked in thermodynamics, so I shall use the symbol γ (although, it must be admitted, γ is heavily worked in thermodynamics, too.)

Not everyone is comfortable with a definition involving forces perpendicular to an imaginary line drawn in the surface, and an alternative approach may be more palatable to some. The idea of a molecule beneath the surface being surrounded on all sides by other molecules and hence experiencing zero net average force, while a molecule on the surface is pulled asymmetrically by the molecules beneath it, remains. But instead of drawing an imaginary line on the surface, we reason that it requires *work* to move a molecule from within the liquid to the surface, and it requires a lot of work to move many molecules from beneath to the surface. That is, it requires work to create new surface. Thus we can define surface tension as the *work required to create unit area of new surface*. The conditions under which this work is done have to be carefully defined in any precise definition, and, from a thermodynamical point of view, the strict definition is the increase in the Gibbs free energy per unit area of new surface created under conditions of constant temperature and pressure. That is:

$$\gamma = \left(\frac{\partial G}{\partial A} \right)_{T,P}$$

20.2.1

This is consistent with the definition of the Gibbs free energy as a quantity whose increase is equal to the work , other than PdV work, done on a system in a reversible, isothermal isobaric process.

Such a nicety will be of interest to those versed in thermodynamics (and I have added a bit about the thermodynamics of surface energy in Chapter 12 of Thermodynamics), but for those not so versed, you may, without any serious prejudice to understanding most of the matter in this section, think of surface tension either as the force per unit length perpendicular to an imaginary line in the surface, or as the work required to create unit area of new surface. You may express surface tension either in newtons per metre or in joules per square metre (or, if you are of CGS persuasion, dynes per centimetre or ergs per square centimetre). These are dimensionally equivalent.

20.2.1 *Excess Pressure Inside Drops and Bubbles*

The pressure inside a spherical drop is greater than the pressure outside. The way in which the excess pressure P depends on the radius a of the drop, and the surface tension γ and density ρ of the liquid is amenable to dimensional analysis. One can suppose that $P \propto a^{\alpha}\gamma^{\beta}\rho^{\delta}$, after which I leave it to the reader to show that $\alpha = -1$, $\beta = 1$, $\delta = 0$, and therefore $P \propto \gamma / a$.

However, it is also quite easy to calculate the excess pressure (other than as a mere proportionality) in terms of the surface tension and the radius of the drop. In figure XX.2 I have divided a spherical

drop of radius a into two hemispheres, and we are going to consider the equilibrium of the upper hemisphere.

The upper hemisphere is being pulled down by surface tension all round the base of the hemisphere, and this downward force is equal to the circumference of the base times the surface tension, or $2\pi\gamma a$.

If the excess pressure inside the drop is P, the upward component of the force due to this pressure is equal to P times the area of the base, πa^2. In case this is not obvious, consider an elemental area dA
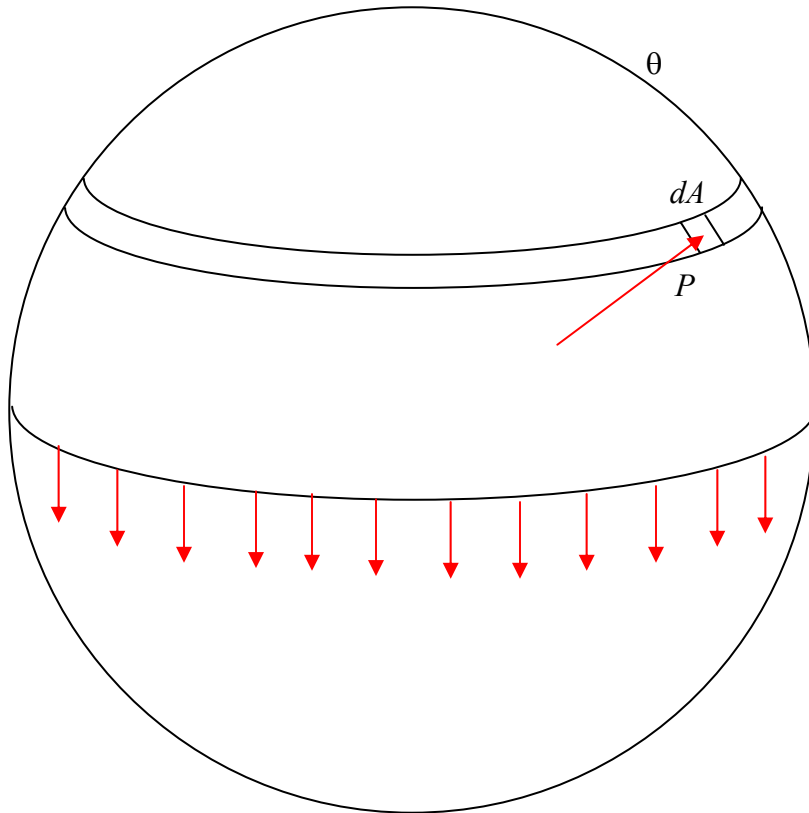

FIGURE XX.2

as shown, at a spherical angle θ from the top of the drop. The force on this element is equal to $P dA$. The upward component of this force is $P \cos \theta \, dA$, and this is equal to P times the horizontal projection of dA. Now you are welcome to do a nice double integration over the hemisphere, but since this (i.e " this is equal to P times the horizontal projection of dA ") is true for every elemental area over the surface of the hemisphere, the total upward force must be equal to P times the area of the base. Thus $2\pi\gamma a = \pi a^2 P$, and so the excess pressure inside the drop is

$$P = \frac{2\gamma}{a} .$$

20.2.2

The smaller the drop, the greater the excess pressure. You may regard this as an explanation as to why droplets cannot form from a vapour unless there is a dust nucleus of finite size for them to condense upon. Of course, two molecules colliding with each other cannot in any case coalesce unless there is something to remove or absorb the kinetic energy.

The case of a nonspherical drop might be mentioned in passing. It is a well known result in geometry (or at least it is well known to those who already know it) that if $z = z(x, y)$ is a nonspherical surface, and you take two vertical planes at right angles to each other, and if a_1 and a_2 are the radii of curvature of the intersections of the two planes with the surface, then $\dfrac{1}{a_1} + \dfrac{1}{a_2}$ is independent of the orientations of the two planes, as long as they remain perpendicular to each other. In other words, a_1 and a_2 do not have to be the maximum and minimum radii of curvature. The excess pressure inside a nonspherical drop is

$$P = \gamma\left(\frac{1}{a_1} + \frac{1}{a_2}\right).$$

20.2.3

What about the pressure inside a spherical bubble of air (or other gas) under water (or other liquid)? If we are hasty, we might suggest that, since this is the opposite situation to a liquid drop in air, maybe the pressure is *less* inside an underwater bubble. This would be a very hasty conclusion, and quite wrong. If you go through exactly the same argument as we did for a drop, considering the equilibrium of one hemisphere, you will see immediately that there is (as for the drop) an *excess* pressure inside the bubble given again by equation 20.2.2. And exactly the same would apply to a spherical drop of one liquid under the surface of a second liquid, if the two liquid are immiscible. But, rather than just repeat the identical derivation, let's try a different approach.

Let us imagine that we have a bubble of radius a in a liquid of surface tension γ, and suppose that we are able, by means of a fine syringe, to inject some more air inside so as to increase the radius of the bubble by da at constant pressure and temperature. The surface area of a sphere of radius a is $A = 4\pi a^2$, so, if we increase the radius by da we increase the surface area by $8\pi a\, da$, and we increase the volume by $4\pi a^2\, da$. The work done against the surface tension is $8\pi\gamma a\, da$, and this must also be equal to $4\pi P a^2\, da$, where P is the excess pressure inside the bubble. Equating these two expressions leads again to equation 20.2.2.

What about a hollow spherical soap bubble in air? Here the soap has *two* surfaces – inside and out. If you repeat either of the above derivations to this case, you will see that the excess pressure inside a hollow spherical soap bubble is

$$P = \frac{4\gamma}{a}.$$

20.2.4

20.2.2 *Angle of Contact*

When a static quantity of liquid is in contact with an impermeable solid surface, it generally rests so that there is a characteristic angle (measured in the liquid) between the surface of the liquid and the surface of the solid. This angle is the *angle of contact*, and is shown as the angle θ is figure XX.3. Figure XX.3(a) shows an acute angle of contact, in which the liquid spreads out a little and "wets" the surface. Figure XX.3(b) shows an obtuse angle of contact, in which the liquid "bunches up", and does not wet the surface, rather like drops of mercury on most surfaces, or drops of water on the surface of a car that has been freshly waxed. In many cases the angle of contact is close to either $0°$ or $180°$, although it will be appreciated that if θ were *exactly* zero, the liquid would spread

(a)   (b)

FIGURE XX.3

out in an infinitesimally thin layer to cover or "wet" the entire surface; and if it were *exactly* $180°$, the liquid, in the absence of other forces (such as its weight!), would form a spherical globule in contact with the surface only at a single point.

The angle of contact is determined by the nature of *both* surfaces, and is very sensitive to any surface contamination. In order to wet a surface, water may need to be helped by a small amount of wetting agent or detergent; only a small amount is necessary, because only the surface, and not the bulk, of the liquid is involved. The *chemical* nature of wetting agents and detergents is beyond the scope of these notes (i.e. it is beyond *my* scope!), but how the angle of contact depends on the surface tension provides a useful example of the technique of *virtual work* (see Section 9.4 of Chapter 9).
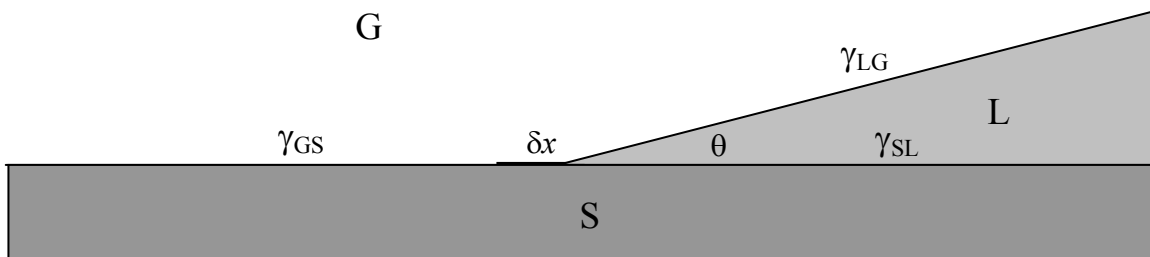

FIGURE XX.4

Figure XX.4 represents a liquid, L, (e.g. water) in contact with a solid, S, (e.g. glass) and a gas, G, (e.g. air). The angle of contact is θ, and the surface tensions (energy per unit area, or, for those who are versed in thermodynamics, the Gibbs free energy per unit area) of the three interfaces are as shown. We'll suppose that the three media extend for a distance l at right angles to the plane of the paper (or computer screen). The three phases are in equilibrium. Now, if we move the SLG boundary to the left by a distance δx, we create a new area $l\delta x$ of SL interface and a new area $l\cos\theta\,\delta x$ of LG interface, while we lose an area $l\delta x$ of GS interface. The work done on the system is therefore $\gamma_{SL}l\delta x + \gamma_{LG}l\cos\theta\delta x - \gamma_{GS}l\delta x$. By the principle of virtual work, this is zero, and therefore

$$\cos\theta = \frac{\gamma_{GS} - \gamma_{SL}}{\gamma_{LG}}.$$

20.2.5

The angle of contact is acute or obtuse, according to whether γ_{GS} is greater than or less than γ_{SL}.

20.2.3 *Capillary Rise*

When the lower end of a narrow capillary tube is immersed in a liquid, the liquid inside the tube rises a little above the level of the liquid outside. If is then very simple to calculate how far the liquid rises in terms of the surface tension, the angle of contact and the inside radius of the tube. See figure XX.5.

FIGURE XX.5

h
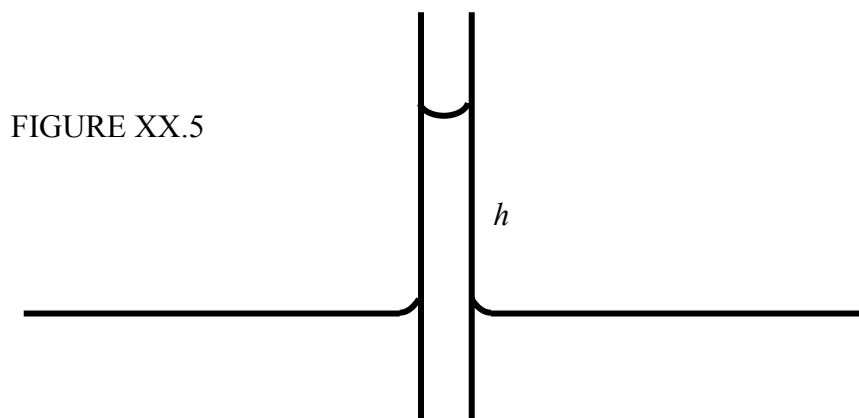
The upward force due to surface tension is $2\pi a\gamma\cos\theta$, where a is the inside radius of the tube, and, if we neglect the very small mass of the liquid in the meniscus (the curved surface at the top of the liquid column), the weight of the liquid column is $\pi a^2 h\rho g$, and therefore

$$h = \frac{2\gamma\cos\theta}{\rho g a}.$$

20.2.6

(Check the dimensions!)

Of course if θ is *obtuse* (as with mercury in contact with glass), *h* will be *negative,* and the level of the mercury in the tube will be *below* the outside level.

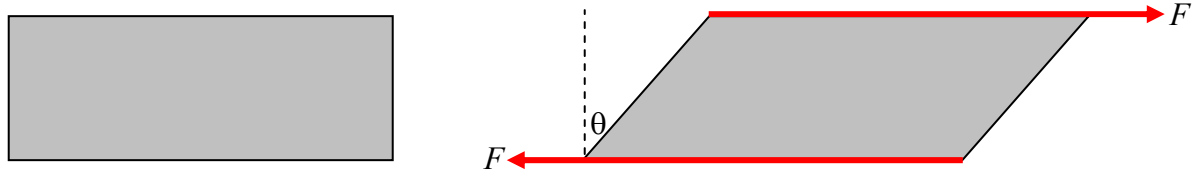### 20.3   *Shear Modulus and Torsion Constant*

FIGURE XX.6

Imagine that we have a rectangular block of solid material, as shown on the left hand side of figure XX.6. We now apply a couple of tangential forces *F* as shown on the right hand side. (I have not decided to go all chatty and informal by saying "a couple" of forces; far from it – I am using the word "couple" in its formal sense in mechanics.)   The material will undergo an angular deformation, and the ratio of the tangential force per unit area to the resulting angular deformation is called the *shear modulus* or the *rigidity modulus*. Its SI unit is $N\,m^{-2}\,rad^{-1}$ and its dimensions are $ML^{-1}T^{-2}\theta^{-1}$.   (I'd advise against using "pascals" per radian. The unit "pascal" is best restricted to pressure, which is normal force per unit area, and is not quite the same thing as the tangential force per unit area that we are discussing here.) You should convince yourself that the definition must specify the force *F*, not the torque provided by the couple. If the block were twice as thick, and the forces were the same, you'd still get the same angular deformation.

If you hold one end of a wire or rod fixed and apply a torque to the other end, this end will twist through an angle, and the ratio of the applied torque to the angle through which the wire twists is the *torsion constant*, *c*, of the wire. You can see how the torsion constant depends on the shear modulus η of the metal and the radius *a* and length *l* of the wire by the method of dimensions. You can start by supposing that $c \propto \eta^{\alpha} a^{\beta} l^{\gamma}$, but you will soon find yourself in difficulty because *a* and *l* are each of dimension L.   However, you will probably have no difficulty with making the assumption that $\gamma = -1$ (the longer the wire, the easier it is to twist), and dimensional analysis will soon show that $\alpha = 1$ and $\beta = 4$ -   which, being interpreted, meaneth that it is much more difficult to twist a thick wire than a thin wire. But can we do better and get an expression other than a mere proportionality for the torsion constant? Can we find the proportionality constant?   Let's try some simpler problems first, and see how things go.

Let us consider a long, thin strip or ribbon of metal. By long and thin I mean that its length is much greater than its width, and its width is much greater than its thickness. I can use any symbol I like to represent any quantity I like, so I could, if I wished, use Ξ for the length, m_{α} for the width, and G_2 for the thickness. Instead, the symbols that I shall choose to represent the length, width and thickness of the strip are going to be, respectively, l, $2\pi r$ and δr. This seems silly at the moment,

but in the end you'll be glad that I made this choice. The strip is shown at the left hand side of figure XX.7.
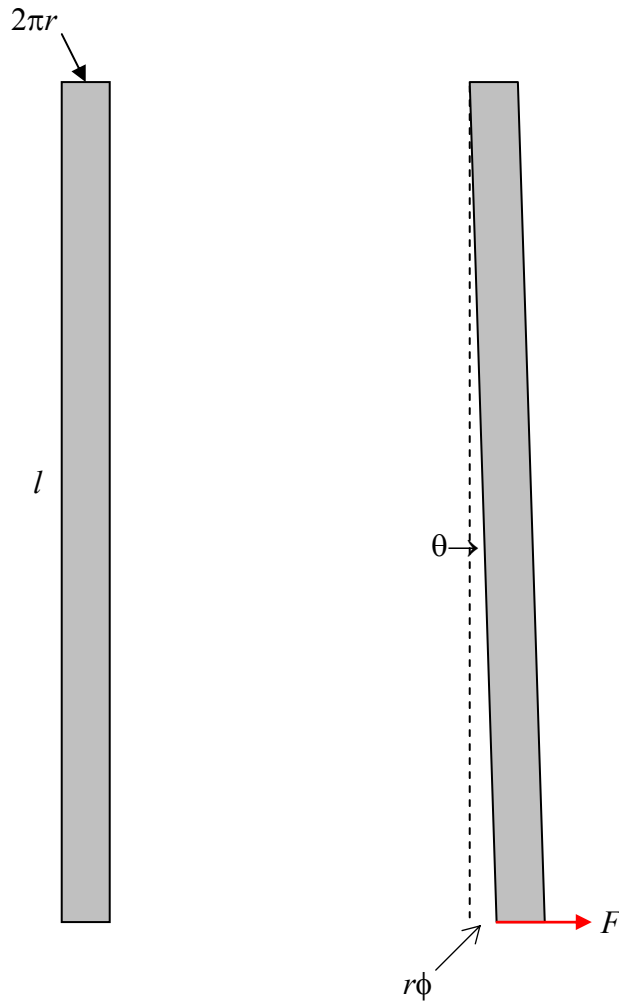

$2\pi r$

l

$\theta \rightarrow$

$r\phi$

F

FIGURE XX.7

I am now going to fix the upper end of the strip and apply a force F to the lower end, as shown in the right hand side of figure XX.7, and I can use any symbol I like to represent the displacement of the lower end, and I choose the symbol $r\phi$. This means that the angular displacement θ is equal to $r\phi/l$. The tangential force per unit area is $F/(2\pi r \, \delta r)$, and therefore

$$\eta = \frac{Fl}{2\pi\phi r^2 \, \delta r} ,$$

20.3.1

or

$$F = \frac{2\pi\eta\phi r^2 \delta r}{l} .$$

20.3.2

Now I'm going to restore the strip to its original shape, and then I'm going to roll it into a hollow cylindrical tube, so that it now looks like a metal drinking straw. The circumference of the straw is $2\pi r$, its radius is r and its thickness is δr (figure XX.8). (Now my notation is beginning to make some sense!)

FIGURE XX.8

I shall hold the upper end of the tube fixed and I shall apply a torque $\tau = Fr$ to the lower end. The tube will evidently twist through an azimuthal angle ϕ given by

$$\tau = \frac{2\pi\eta r^3\delta r}{l}\phi \; .$$
20.3.3

The torsion constant of the hollow tube is therefore

$$c = \frac{2\pi\eta r^3\delta r}{l} \; .$$
20.3.4

The torsion constant of a long solid cylinder (a wire) of radius a is the integral of this from 0 to a, which is

$$c = \frac{\pi \eta a^4}{2l}.$$

20.3.5

20.4 *Viscosity*

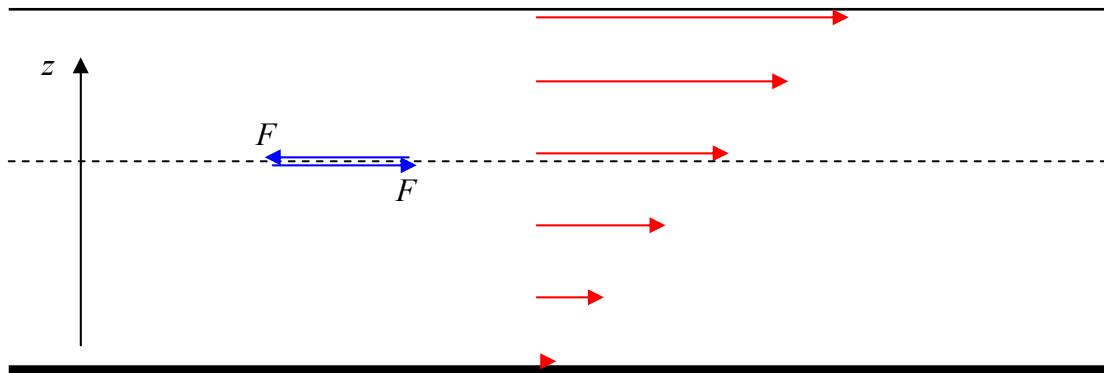Consider a river flowing over a smooth bed, as in figure XX.9.

FIGURE XX.9

There will be a *transverse velocity gradient dv/dz*, with the liquid stationary at the river bottom, and the speed becoming faster as we ascend from the bottom. As a consequence of the transverse velocity gradient, the liquid below the dashed line will be dragged forward by the tangential force of the faster liquid above it, and the liquid above the dashed line will be dragged backward by the tangential force of the more sluggish liquid below it. The ratio of the tangential force per unit area to the transverse velocity gradient is called the coefficient of *dynamic viscosity*, for which the usual symbol is η. The dimensions of dynamic viscosity are $ML^{-1}T^{-1}$. The CGS unit of dynamic viscosity is the *poise.* The abbreviation for the unit is P – though it would be well to define it if you use it, since not everyone will recognize it. The unit is named after a nineteenth century French doctor, Jean Poiseuille, who was interested in blood pressure and hence in the rate of flow of liquids through tubes.   That is to say, if, for a transverse velocity gradient of 1 cm s^{-1} per cm, the tangential force per unit area is 1 dyne cm^{-2}, the dynamic viscosity is one poise. The SI (MKS) unit is the decapoise (also spelled dekapoise), though the SI unit the pascal second (Pa s), which is dimensionally correct, is also seen. If, for a transverse velocity gradient of 1 m s^{-1} per cm, the tangential force per unit area is 1 N m^{-2}, the dynamic viscosity is one decapoise.   The dynamic viscosity of water varies from about 1.8 centipoise at 0°C to about 0.3 centipoise at 100°C.

The ratio of the coefficient of dynamic viscosity to the density is the coefficient of *kinematic viscosity,* for which the usual symbol is the Greek letter ν. The dimensions of kinematic viscosity are L^2T^{-1}. The CGS unit of kinematic viscosity is the *stokes* (abbreviation St). It is named after nineteenth century British physicist, Sir George Stokes, who made major contributions to diverse areas of physics. The SI unit of kinematic viscosity is usually given simply as $m^2\ s^{-1}$, and $1\ m^2\ s^{-1}$ $= 10^4$ stokes. The kinematic viscosity of water varies from about 1.8 centistokes ($1.8 \times 10^{-6}\ m^2\ s^{-1}$) at 0°C to about 0.3 centistokes ($3 \times 10^{-7}\ m^2\ s^{-1}$) at 100 °C.

Hydrodynamics is a huge and very difficult subject (at least I think it is), but there are a couple of simple problems that, if nothing else, make good homework problems.   These are Poiseuille's law and the Couette viscometer.

20.4.1 *Poiseuille's Law*

Poiseuille's law tells you how the rate of nonturbulent flow of a liquid through a cylindrical pipe depends on the viscosity of the liquid, the radius of the pipe, and the pressure gradient. If all else fails, you can at least try dimensional analysis. Assume that the rate of flow of liquid (in cubic metres per second) is proportional to $\eta^{\alpha} a^{\beta}\left(\dfrac{dP}{dx}\right)^{\gamma}$, and show by dimensional analysis that $\alpha = -1$, $\beta = -4$ and $\gamma = 1$, which shows that the rate of flow is very sensitive to the radius of the pipe. That $\beta = -4$ tells you that if your arteries are at all constricted, even by a little bit, you had better watch out. Gas flow is more complicated because gases are compressible, (so are liquids, but not by much), but $\beta = -4$ tells you that the rate at which you can pump out gas from a system depends a lot on the size of the smallest tube you have between the volume that you are trying to evacuate and the pump. Now let's try and analyse it further.
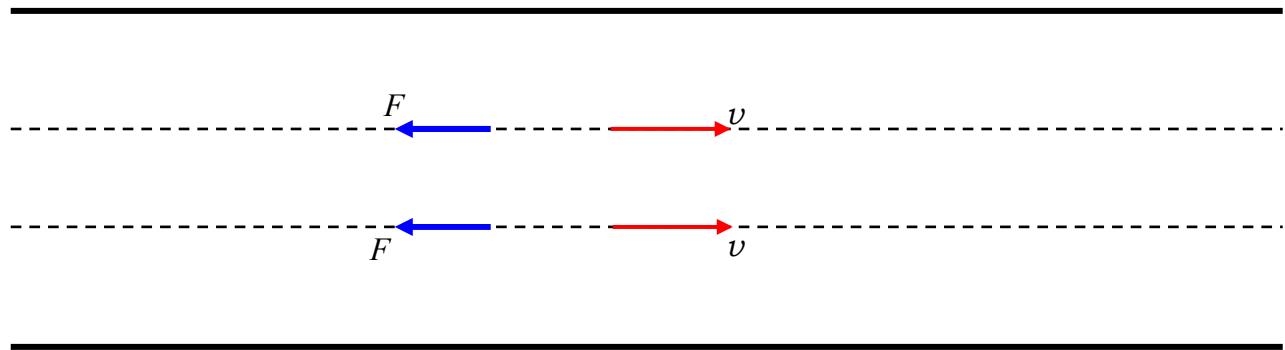

FIGURE XX.10

Figure XX.10 represents a pipe of radius *a* with liquid flowing to the right. At a distance *r* from the axis of the pipe the speed of the liquid is *v.* The length of the pipe is *l*, and there is a pressure gradient along the length of the pipe, the pressure at the left end being higher than the pressure at

the right by P. There is a velocity gradient in the pipe. The speed of the liquid along the axis of the pipe is v_0, and the speed at the circumference of the pipe is zero. That is, the speed decreases from axis to circumference, so that the velocity gradient (dv/dr) is negative.

Now consider the equilibrium of the liquid inside radius r. (It *is* in equilibrium because it is moving at constant speed.) It is being pushed forward by the pressure gradient. This rightward force is $\pi r^2 P$. It is being dragged back by the viscous force acting on the area $2\pi rl$. This leftward force is $-2\pi\eta lr(dv/dr)$, this expression for the leftward force being positive.

Therefore
$$-2\eta l\frac{dv}{dr} = Pr. \qquad\qquad 20.4.1$$

Integrate from the axis ($r = 0$, $v = v_0$) to r:

$$v = v_0 - \frac{Pr^2}{4\eta l}. \qquad\qquad 20.4.2$$

Thus the speed decreases quadratically (parabolically) as you move away from the axis. The speed is zero at the circumference, and hence the speed on the axis is

$$v_0 = \frac{Pr^2}{4\eta l}. \qquad\qquad 20.4.3$$

Verify the dimensions.

Now the volume flow through a cylindrical shell of radii r and $r + dr$ is the speed times the area $2\pi r\,dr$, which is $\dfrac{\pi Pr^3 dr}{2\eta l}$, and if you integrate that through the whole pipe, from 0 to a, you find that the rate of flow of liquid through the pipe (cubic metres per second) is

$$\frac{\pi a^4 P}{8\eta l}. \qquad\qquad 20.4.4$$

This is *Poiseuille's Law*.

20.4.2 *The Couette Viscometer*

A cylindrical vessel of radius b contains the liquid whose viscosity is to be measured. A smaller, solid cylinder of radius a and length l is suspended from a torsion wire, whose torsion constant c is known, and is immersed in the liquid in the vessel, the two cylinders being coaxial. The vessel containing the liquid is spun about its axis at an angular speed Ω, thus setting the liquid in motion. This causes a viscous torque on the inner cylinder, which is therefore pulled round through an angle

ϕ. When the restoring torque of the torsion wire $c\phi$ is equal to the viscous torque, the system will be in equilibrium, and one can then calculate the viscosity η of the liquid. We shall refer to figure XX.11. In the simple analysis given below, we suppose that the angular and linear speed and gradients are sufficiently small that the flow is nonturbulent. We also neglect the effects of viscous drag on the flat ends of the cylinder. Thus the diameter of the cylinder, in our analysis, must be much less than its length.

Incidentally, for a long time I thought that the word "couette" must be French for something. It is – it's French for "feather bed" or for "pigtail". But the Couette viscometer is actually named after a little-known nineteenth century French scientist, Maurice Couette.
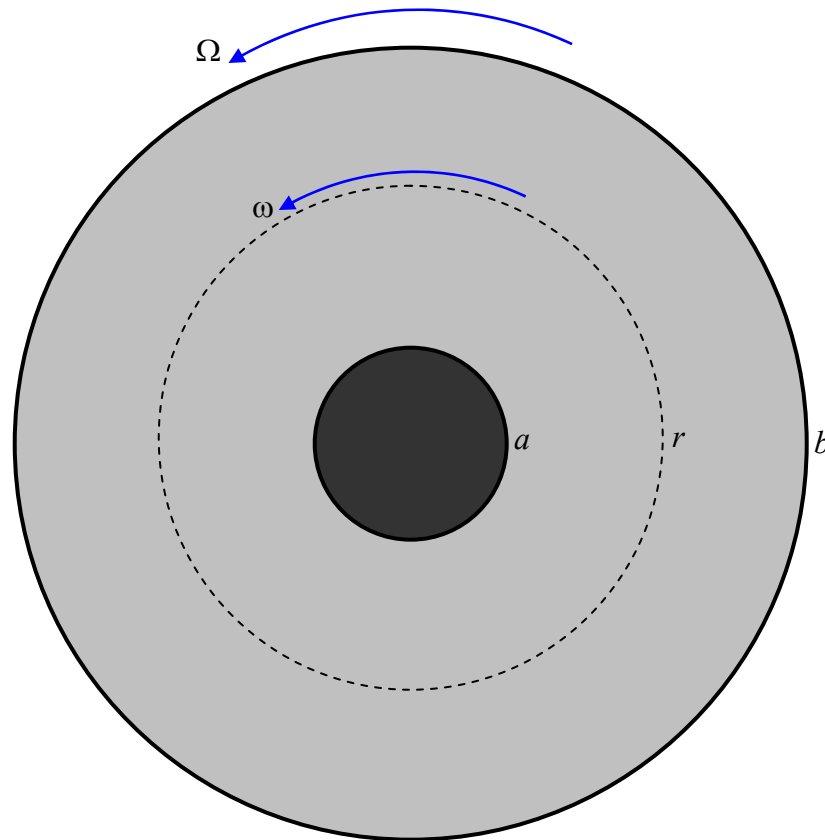

FIGURE XX.11

Let us calculate the viscous torque on the liquid within radius r. Notice that, since we have a steady-state situation, this torque is *independent of r*; in particular the torque on the liquid within radius r is the same as the torque (which we can measure with the torsion wire) on the inner cylinder. The area of the curved surface of the liquid within radius r is $2\pi lr$. The viscous torque on this surface is r times η times the area times the transverse velocity gradient. But we have to be careful about this last term. If the whole body of the liquid were rotating as a solid body with angular speed ω, the speed at radius r would be $r\omega$ and hence there would be a transverse velocity gradient equal to ω – but no viscous drag! But the liquid is not, of course, rotating as a solid, and ω

(as well as v) is a function of r. Since $v = r\omega$, the velocity gradient is $\dfrac{dv}{dr} = r\dfrac{d\omega}{dr} + \omega$, and the only part of this that goes into the expression for the viscous torque is the part $r\dfrac{d\omega}{dr}$. Thus the expression for the torque on the liquid within radius r (and hence also on the inner cylinder) is

$$\tau = r.\eta.2\pi rl.r\frac{d\omega}{dr}.\qquad\qquad\text{20.4.5}$$

That is,
$$\frac{d\omega}{dr} = \frac{\tau}{2\pi\eta lr^3}.\qquad\qquad\text{20.4.6}$$

Integration from $r = a$, $\omega = 0$ to $r = b$, $\omega = \Omega$ gives

$$\tau = \frac{4\pi\eta l\Omega a^2 b^2}{b^2 - a^2}.\qquad\qquad\text{20.4.7}$$

In equilibrium, this is equal to $c\phi$, where c is the torsion constant of the suspension and ϕ is the angle through which the inner cylinder has turned, and hence the viscosity can be determined. You should, as usual, check the dimensions of equation 20.4.7.