CHAPTER 2
MOMENT OF INERTIA

2.1 *Definition of Moment of Inertia*

Consider a straight line (the "axis") and a set of point masses $m_1, m_2, m_3, \dots$ such that the distance of the mass m_i from the axis is r_i. The quantity $m_i r_i^2$ is the second moment of the i th mass with respect to (or "about") the axis, and the sum $\sum m_i r_i^2$ is the second moment of mass of all the masses with respect to the axis.

Apart from some subtleties encountered in general relativity, the word "inertia" is synonymous with mass - the inertia of a body is merely the ratio of an applied force to the resulting acceleration. Thus $\sum m_i r_i^2$ can also be called the *second moment of inertia*. The second moment of inertia is discussed so much in mechanics that it is usually referred to as just "the" moment of inertia.

In this chapter we shall consider how to calculate the (second) moment of inertia for different sizes and shapes of body, as well as certain associated theorems. But the question should be asked: "What is the purpose of calculating the squares of the distances of lots of particles from an axis, multiplying these squares by the mass of each, and adding them all together? Is this merely a pointless make-work exercise in arithmetic? Might one just as well, for all the good it does, calculate the sum $\sum m_i^2 r_i$? Does $\sum m_i r_i^2$ have any physical significance?"

2.2 *Meaning of Rotational Inertia*.

If a force acts of a body, the body will accelerate. The ratio of the applied force to the resulting acceleration is the inertia (or mass) of the body.

If a torque acts on a body that can rotate freely about some axis, the body will undergo an angular acceleration. *The ratio of the applied torque to the resulting angular acceleration is the rotational inertia* of the body. It depends not only on the mass of the body, but also on how that mass is distributed with respect to the axis.

Consider the system shown in figure II.1.

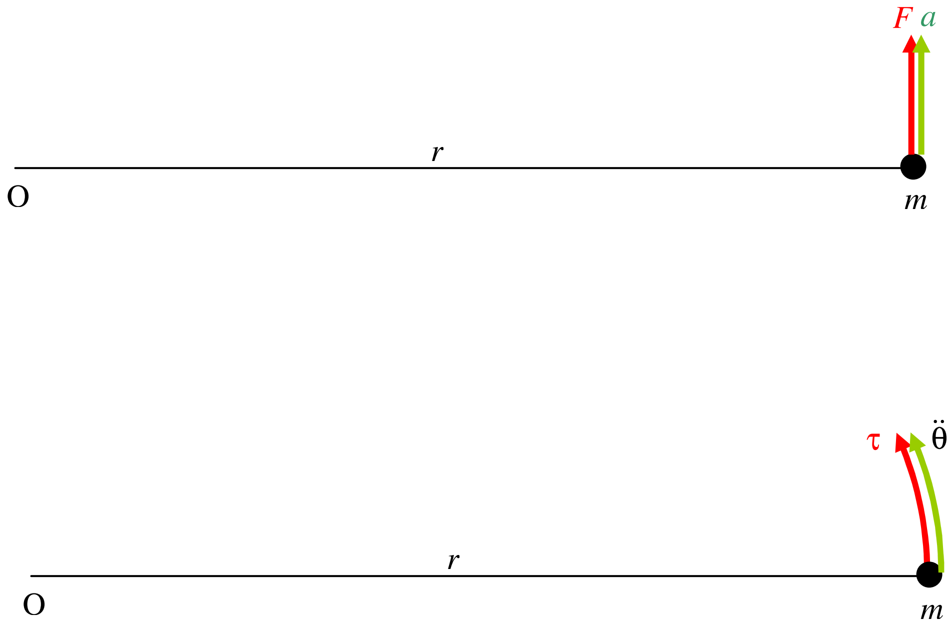


FIGURE II.1

A particle of mass *m* is attached by a light (i.e. zero or negligible mass) arm of length *r* to a point at O, about which it can freely rotate. A force *F* is applied, and the mass consequently undergoes a linear acceleration $a = F/m$. The angular acceleration is then $\ddot{\theta} = F.mr$. Also, the torque is $\tau = Fr$. The ratio of the applied torque to the angular acceleration is therefore mr^2. Thus the rotational inertia is the second moment of inertia. Rotational inertia and (second) moment of inertia are one and the same thing, except that rotational inertia is a physical concept and moment of inertia is its mathematical representation.

2.3 *Moments of inertia of some simple shapes.*

A student may well ask: "For how many different shapes of body must I commit to memory the formulas for their moments of inertia?" I would be tempted to say: "None". However, if any are to be committed to memory, I would suggest that the list to be memorized should be limited to those few bodies that are likely to be encountered very often (particularly if they can be used to determine quickly the moments of inertia of other bodies) and for which it is easier to remember the formulas than to derive them. With that in mind I would recommend learning no more than five. In the following, each body is supposed to be of mass *m* and rotational inertia *I*.

1. A rod of length 2*l* about an axis through the middle, and at right angles to the rod:

$$I = \tfrac{1}{3} m l^2 \qquad 2.3.1$$

2. A uniform circular disc of radius *a* about an axis through the centre and perpendicular to the

plane of the disc:

$$I = \tfrac{1}{2} ma^2 \qquad 2.3.2$$

3. A uniform right-angled triangular lamina about one of its shorter sides - i.e. not the hypotenuse. The other not-hypotenuse side is of length a:

$$I = \tfrac{1}{6} ma^2 \qquad 2.3.3$$

4. A uniform solid sphere of radius a about an axis through the centre.

$$I = \tfrac{2}{5} ma^2 \qquad 2.3.4$$

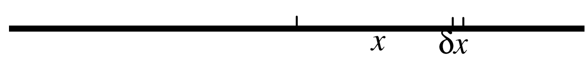
5. A uniform spherical shell of radius a about an axis through the centre.

$$I = \tfrac{2}{3} ma^2 \qquad 2.3.5$$

I shall now derive the first three of these by calculus. The derivations for the spheres will be left until later.

1. Rod, length $2l$ (Figure II.2)

FIGURE II.2



The mass of an element δx at a distance x from the middle of the rod is

$$\frac{m\delta x}{2l}$$

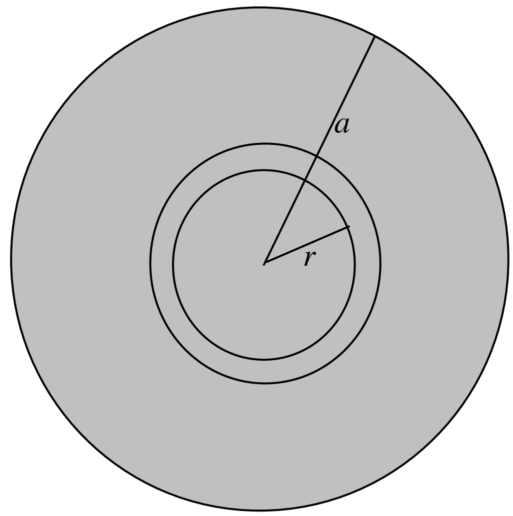
and its second moment of inertia is $\dfrac{mx^2\delta x}{2l}$.

The moment of inertia of the entire rod is

$$\frac{m}{2l}\int_{-l}^{l} x^2 dx = \frac{m}{l}\int_0^l x^2 dx = \tfrac{1}{3} ml^2.$$

2. Disc, radius a. (Figure II.3)

FIGURE II.3



The area of an elemental annulus, radii $r,\quad r+\delta r$ is $2\pi r\delta r$.

The area of the entire disc is πa^2.

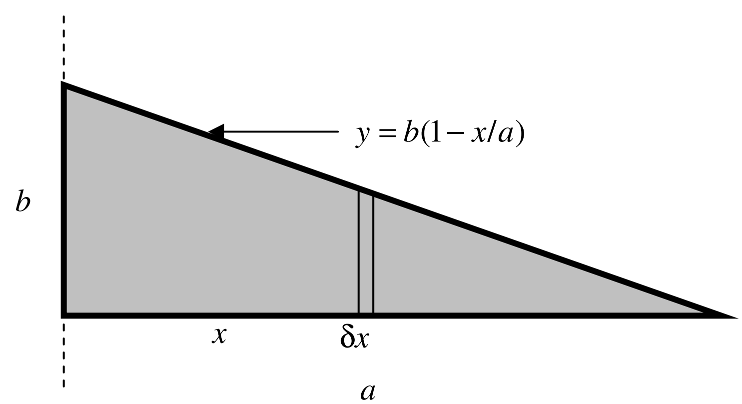
Therefore the mass of the annulus is $\dfrac{2\pi r\delta r m}{\pi a^2} = \dfrac{2mr\delta r}{a^2}$.

and its second moment of inertia is $\dfrac{2mr^3\delta r}{a^2}$.

The moment of inertia of the entire disc is $\dfrac{2m}{a^2}\int_0^a r^3 dr = \tfrac{1}{2}ma^2$.

3. Right-angled triangular lamina. (Figure II.4)

FIGURE II.4



The equation to the hypotenuse is $y = b(1 - x/a)$.

The area of the elemental strip is $y\delta x = b(1 - x/a)\delta x$ and the area of the entire triangle is $ab/2$.

Therefore the mass of the elemental strip is $\dfrac{2m(a-x)\delta x}{a^2}$

and its second moment of inertia is $\dfrac{2mx^2(a-x)\delta x}{a^2}$.

The second moment of inertia of the entire triangle is the integral of this from $x = 0$ to $x = a$, which is $ma^2/6$.

Uniform circular lamina about a diameter.

For the sake of one more bit of integration practice, we shall now use the same argument to show that the moment of inertia of a uniform circular disc about a diameter is $ma^2/4$. However, we shall see later that it is not necessary to resort to integral calculus to arrive at this result, nor is it necessary to commit the result to memory. In a little while it will become immediately apparent and patently obvious, with no calculation, that the moment of inertia must be $ma^2/4$. However, for the time being, let us have some more calculus practice. See figure II.5.

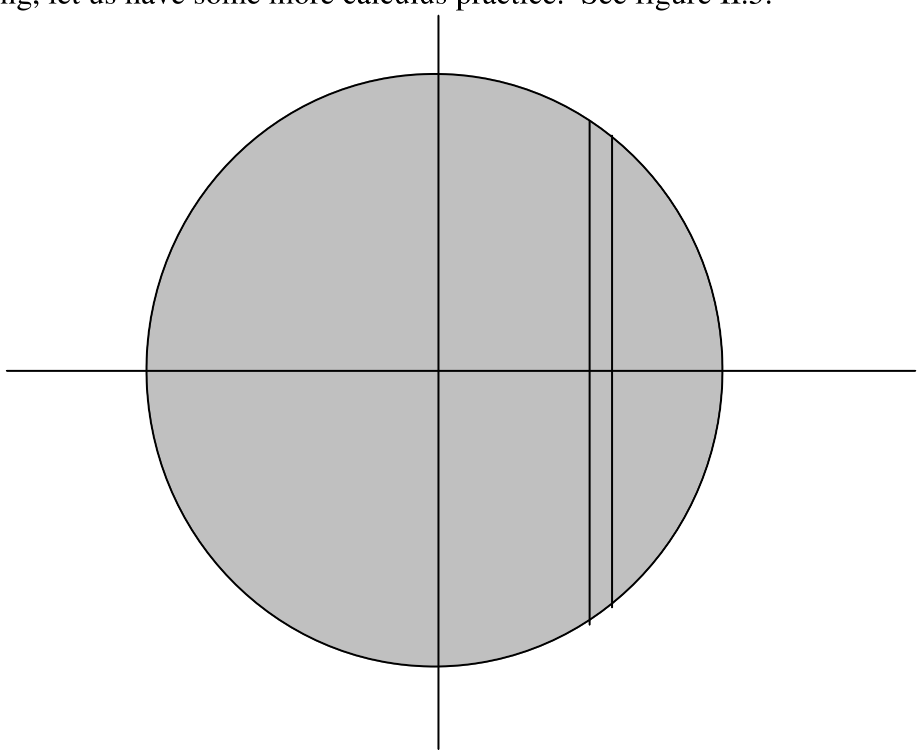


FIGURE II.5

The disc is of radius a, and the area of the elemental strip is $2y\delta x$. But y and x are related through the equation to the circle, which is $y = \left(a^2 - x^2\right)^{1/2}$. Therefore the area of the strip is $2\left(a^2 - x^2\right)^{1/2}\delta x$. The area of the whole disc is πa^2, so the mass of the strip is $m \times \frac{2(a^2 - x^2)^{1/2}\delta x}{\pi a^2} = \frac{2m}{\pi a^2} \times (a^2 - x^2)^{1/2}\delta x$. The second moment of inertia about the y-axis is $\frac{2m}{\pi a^2} \times x(a^2 - x^2)^{1/2}\delta x$. For the entire disc, we integrate from $x = -a$ to $x = +a$, or, if you prefer, from $x = 0$ to $x = a$ and then double it. The result $ma^2/4$ should follow. If you need a hint about how to do the integration, let $x = a\cos\theta$ (which it is, anyway), and be sure to get the limits of integration with respect to θ right.

The moment of inertia of a uniform *semi*circular lamina of mass m and radius a about its base, or diameter, is also $ma^2/4$, since the mass distribution with respect to rotation about the diameter is the same.

2.4 *Radius of gyration.*

The second moment of inertia of any body can be written in the form mk^2. Thus, for the rod, the disc (about an axis perpendicular to its plane), the triangle and the disc (about a diameter), k has the values

$$\frac{l}{\sqrt{3}} = 0.866l, \quad \frac{a}{\sqrt{2}} = 0.707a, \quad \frac{a}{\sqrt{6}} = 0.408a, \quad \frac{a}{2} = 0.500a$$

respectively.

k is called the radius of gyration. If you were to concentrate all the mass of a body at its radius of gyration, its moment of inertia would remain the same.

2.5 *Parallel and Perpendicular Axes Theorems*

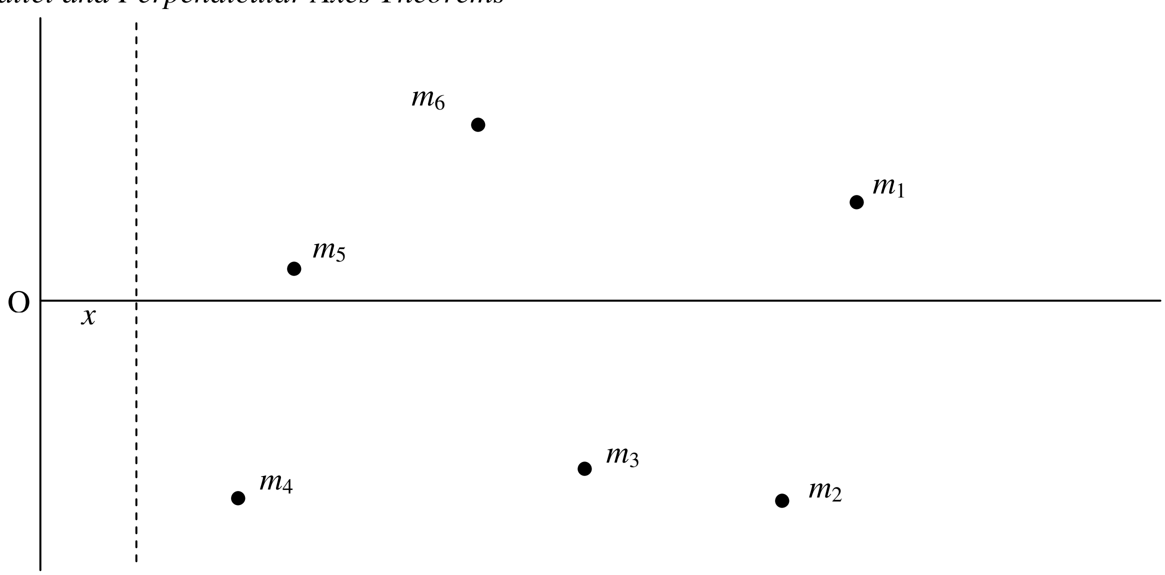


FIGURE II.6a

In figure II.6a, the two unbroken lines represent two fixed coordinate axes. I have drawn several point masses $m_1, m_2, m_3 \ldots$ distributed in a plane. The x-coordinate of mass m_i is x_i. The dashed line is moveable, and it x-coordinate is x, so that the distance of m_i this line is $x_i - x$. The moment of inertia of the system of masses about the dashed line is

$$I = m_1(x_1 - x)^2 + m_2(x_2 - x)^2 + m_3(x_3 - x)^2 + \ldots . \qquad 2.5.1$$

Now imagine what happens if the dashed line is moved to the right. The moment of inertia decreases – and decreases - and decreases. But eventually the line finds itself to the right of m_4, and then of m_5, and then of m_6. After that is by no means obvious that the moment of inertia is going to continue to decrease. Indeed, by this time it is clear that at some point I is going to go through a minimum and then start to increase again as more and more of the masses find themselves to the left of the dashed line. Just where is the dashed line when the moment of inertia is a minimum? I'll leave you to differentiate equation 2.5.1 with respect to x, and hence show that I is least when

$$x = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} . \qquad 2.5.2$$

That is, the moment of inertia is least when $x = \bar{x}$. That is, the moment of inertia is least for an axis passing through the centre of mass.

In figure II.6b, the line CC passes through the centre of mass; the moment of inertia is least about this line. The line AA is at a distance $\bar{x}$ from CC, and the moment of inertia is greater about AA than about CC. The *Parallel Axes Theorem* tells us by how much.

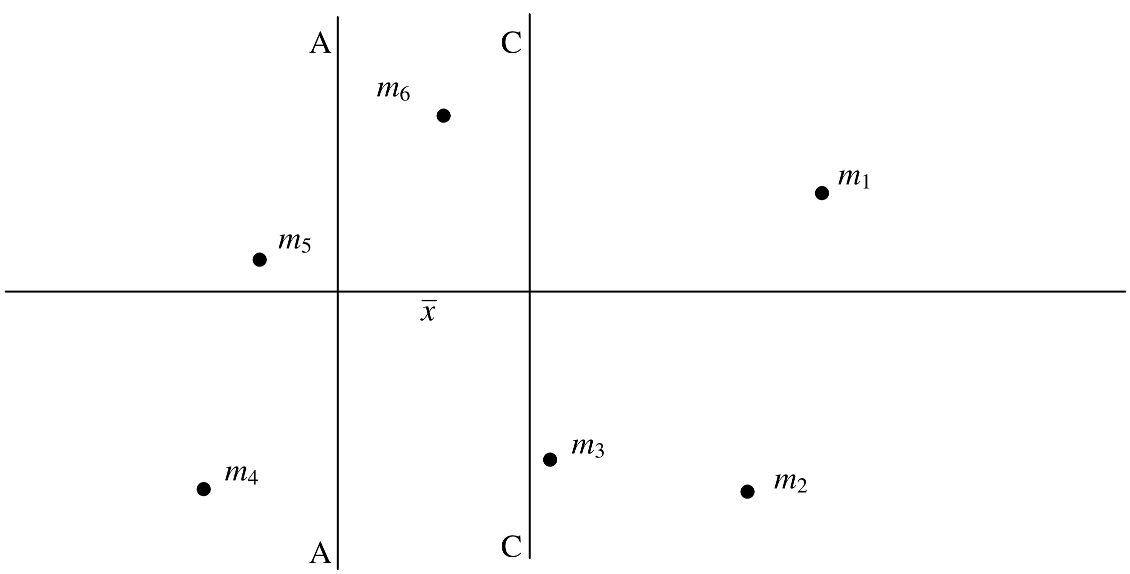


FIGURE II.6b

Let us measure distances from CC, so that the distance of m_i from CC is x_i and the distance of m_i from AA is $x_i + \overline{x}$.

It is clear that $$I_{\rm CC} = \sum m_i x_i^2$$

and that $$I_{\rm AA} = \sum m_i (x_i + \overline{x})^2 = \sum m_1 x_i^2 + 2\overline{x}\sum m_i x_i + \overline{x}^2 \sum m_i . \qquad 2.5.3$$

The first term on the right hand side is $I_{\rm CC}$. The sum in the second term is the first moment of mass about the centre of mass, and is zero. The sum in the third term is the total mass. We therefore arrive at the *Parallel Axes Theorem*:

$$I_{\rm AA} = I_{\rm CC} + M\overline{x}^2 . \qquad 2.5.4$$

In words, the moment of inertia about an arbitrary axis is equal to the moment of inertia about a parallel axis through the centre of mass plus the total mass times the square of the distance between the parallel axes. The theorem holds also for masses distributed in three-dimensional space.

The *Perpendicular Axes Theorem*, on the other hand, *holds only for masses distributed in a plane, or for plane laminas*.

FIGURE II.7

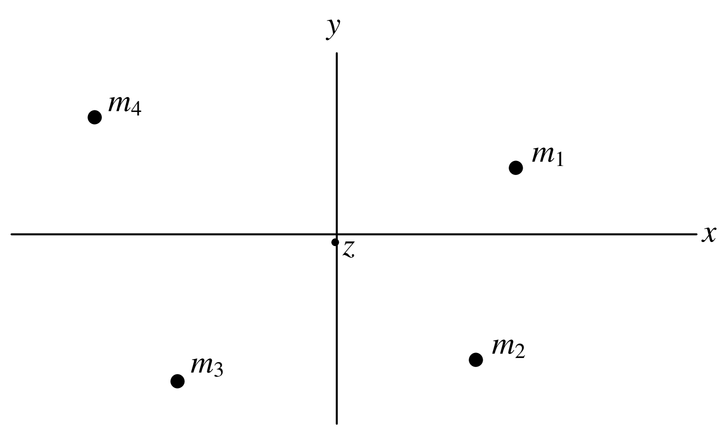


Figure II.7 shows some point masses distributed in the *xy* plane, the *z* axis being perpendicular to the plane of the paper. The moments of inertia about the *x, y* and *z* axes are denoted respectively by *A*, *B* and *C*. The distance of m_i from the *z* axis is $\left(x_i^2 + y_i^2\right)^{1/2}$. Therefore the moment of inertia of the masses about the *z* axis is

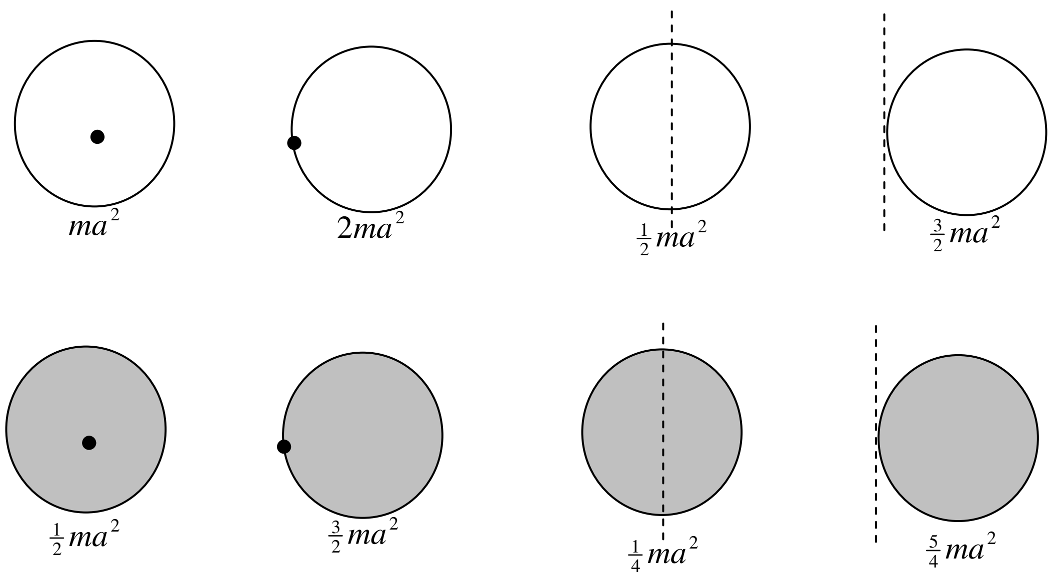
$$C = \sum m_i \left(x_i^2 + y_i^2\right). \qquad 2.5.5$$

That is to say: $$C = A + B. \qquad 2.5.6$$

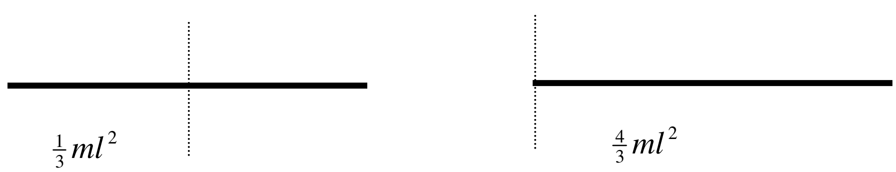
This is the *Perpendicular Axes Theorem*. Note again very carefully that, unlike the parallel axes theorem, this theorem applies only to plane laminas and to point masses distributed in a plane.

Examples of the Use of the Parallel and Perpendicular Axes Theorems.
From section 2.3 we know the moments of inertia of discs, rods and triangular laminas. We can make use of the parallel and perpendicular axes theorems to write down the moments of inertia of most of the following examples almost by sight, with no calculus.

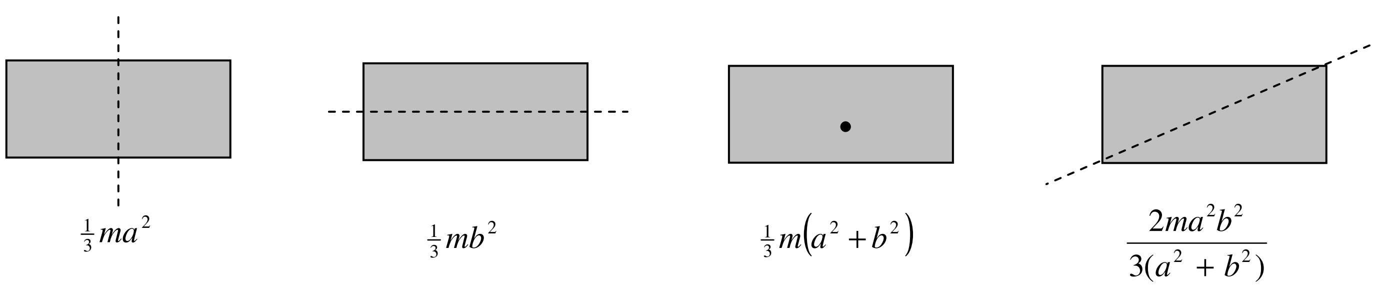
Hoop and discs, radius a.



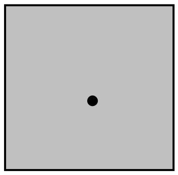
Rods, length $2l$.



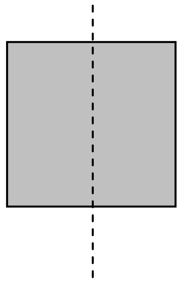
Rectangular laminas, sides $2a$ and $2b$; $a > b$.



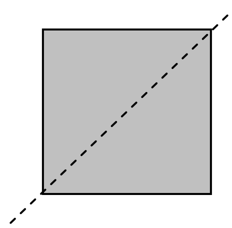
Square laminas, side $2a$.



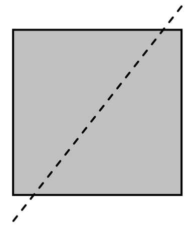
$\frac{2}{3}ma^2$



$\frac{1}{3}ma^2$

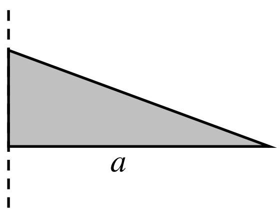


$\frac{1}{3}ma^2$

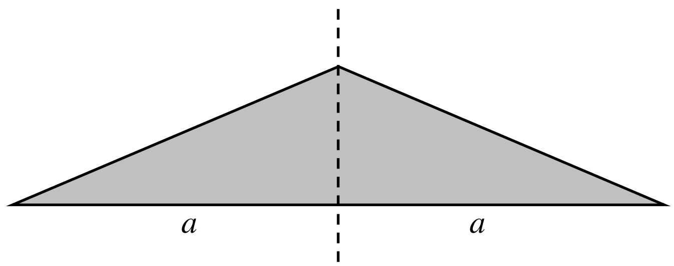


$\frac{1}{3}ma^2$

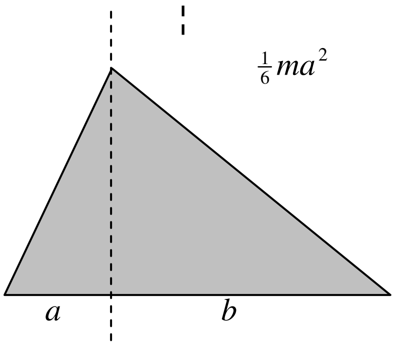
Triangular laminas.



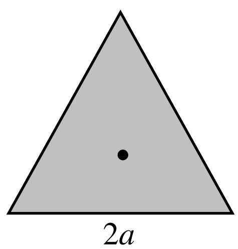
$\frac{1}{6}ma^2$



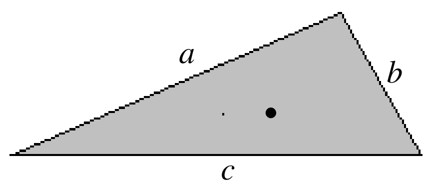
$\frac{1}{6}ma^2$



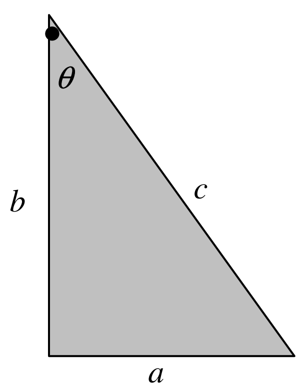
$$\frac{m(a^3+b^3)}{6(a+b)}$$



$\frac{1}{3}ma^2$



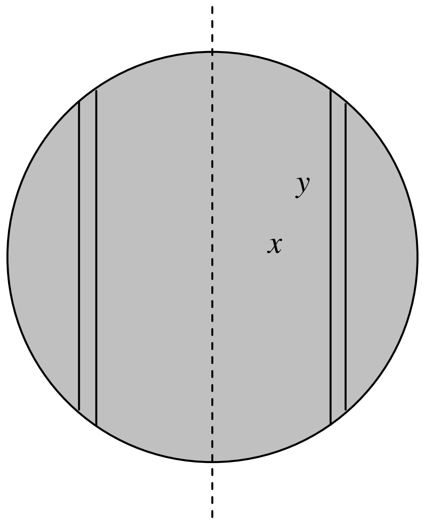
$\frac{1}{36}m\left(a^2+b^2+c^2\right)$



$$I = \tfrac{1}{6}ma^2\left(1+3\cot^2\theta\right) = \tfrac{1}{6}mb^2\left(3+\tan^2\theta\right) = \tfrac{1}{6}mc^2\left(3-2\sin^2\theta\right)$$
$$= \tfrac{1}{6}m\left(2b^2+c^2\right) = \tfrac{1}{6}m\left(3c^2-2a^2\right) = \tfrac{1}{6}m\left(a^2+3b^2\right)$$

2.6 *Three-dimensional solid figures. Spheres, cylinders, cones*

Sphere, mass m, radius a.



The volume of an elemental cylinder of radii x, $x+\delta x$, height $2y$ is $4\pi yx\delta x = 4\pi\left(a^2-x^2\right)^{1/2}x\delta x$. Its mass is $m\times\dfrac{4\pi\left(a^2-x^2\right)^{1/2}x\delta x}{\frac{4}{3}\pi a^3} = \dfrac{3m}{a^3}\times\left(a^2-x^2\right)^{1/2}x\delta x$. Its second moment of inertia is $\dfrac{3m}{a^3}\times\left(a^2-x^2\right)^{1/2}x^3\delta x$. The second moment of inertia of the entire sphere is

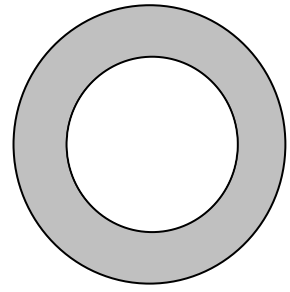
$$\frac{3m}{a^3}\times\int_0^a\left(a^2-x^2\right)^{1/2}x^3\,dx = \tfrac{2}{5}ma^2.$$

The moment of inertia of a uniform solid hemisphere of mass m and radius a about a diameter of its base is also $\tfrac{2}{5}ma^2$, because the distribution of mass around the axis is the same as for a complete sphere.

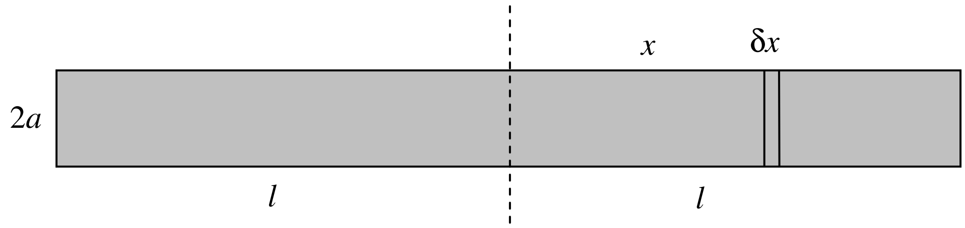
Problem: A hollow sphere is of mass M, external radius a and internal radius xa. Its rotational inertia is 0.5 Ma^2. Show that x is given by the solution of

$$1-5x^3+4x^5=0$$

and calculate x to four significant figures. (Answer = 0.6836.)



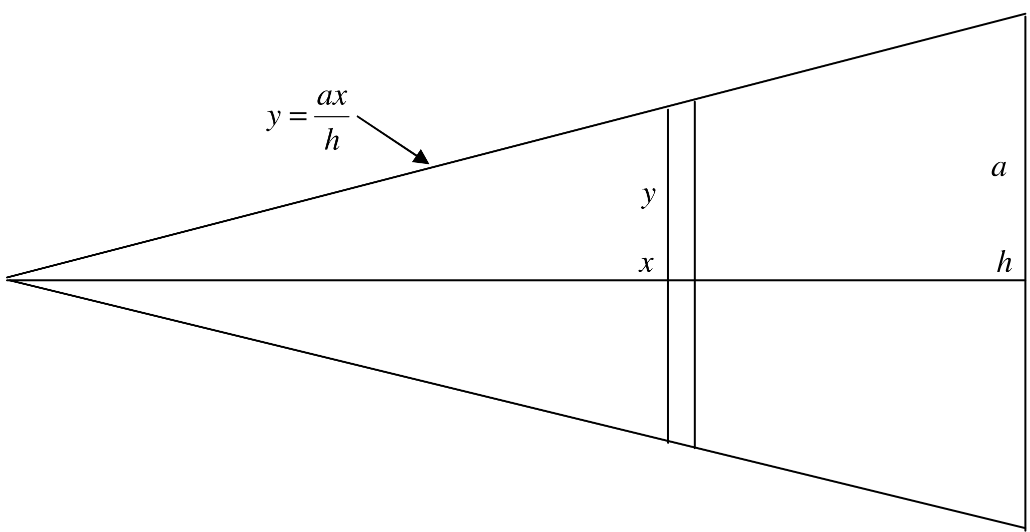
Solid cylinder, mass m, radius a, length 2l



The mass of an elemental disc of thickness δx is $\frac{m\delta x}{2l}$. Its moment of inertia about its diameter is $\frac{1}{4}\frac{m\delta x}{2l}a^2 = \frac{ma^2\delta x}{8l}$. Its moment of inertia about the dashed axis through the centre of the cylinder is $\frac{ma^2\delta x}{8l} + \frac{m\delta x}{2l}x^2 = \frac{m\left(a^2+4x^2\right)\delta x}{8l}$. The moment of inertia of the entire cylinder about the dashed axis is $2\int_0^l \frac{m\left(a^2+4x^2\right)dx}{8l} = m\left(\frac{1}{4}a^2+\frac{1}{3}l^2\right)$.

In a similar manner it can be shown that the moment of inertia of a uniform solid triangular prism of mass m, length $2l$, cross section an equilateral triangle of side $2a$ about an axis through its centre and perpendicular to its length is $m\left(\frac{1}{6}a^2+\frac{1}{3}l^2\right)$.

Solid cone, mass m, height h, base radius a.



The mass of the elemental disc of thickness δx is

$$m \times \frac{\pi y^2 \delta x}{\frac{1}{3}\pi a^2 h} \quad = \quad \frac{3my^2\delta x}{a^2 h}.$$

Its second moment of inertia about the axis of the cone is

$$\frac{1}{2} \times \frac{3my^2\delta x}{a^2 h} \times y^2 \quad = \quad \frac{3my^4\delta x}{2a^2 h}.$$

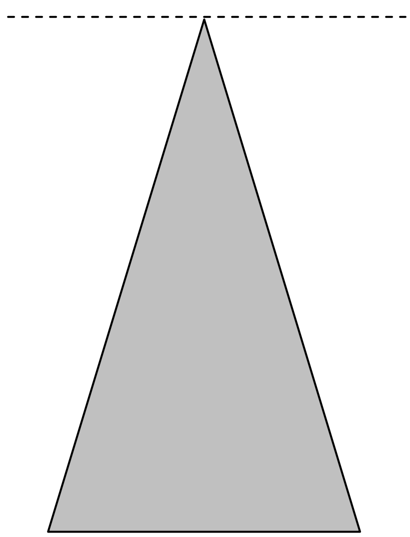
But y and x are related through $y = \dfrac{ax}{h}$, so the moment of inertia of the elemental disk is

$$\frac{3ma^2x^4\delta x}{2h^5}.$$

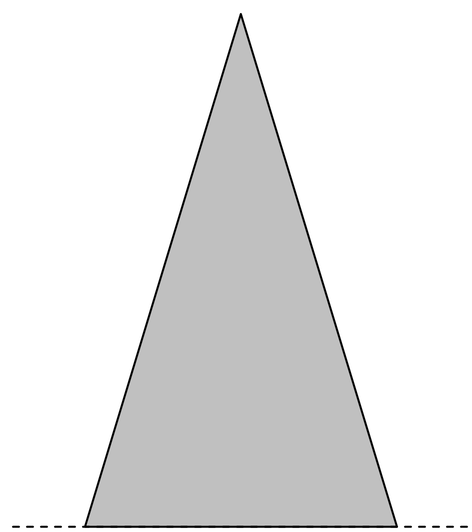
The moment of inertia of the entire cone is

$$\frac{3ma^2}{2h^5}\int_0^h x^4 dx = \frac{3ma^2}{10}.$$

The following, for a solid cone of mass *m*, height *h*, base radius *a*, are left as an exercise:



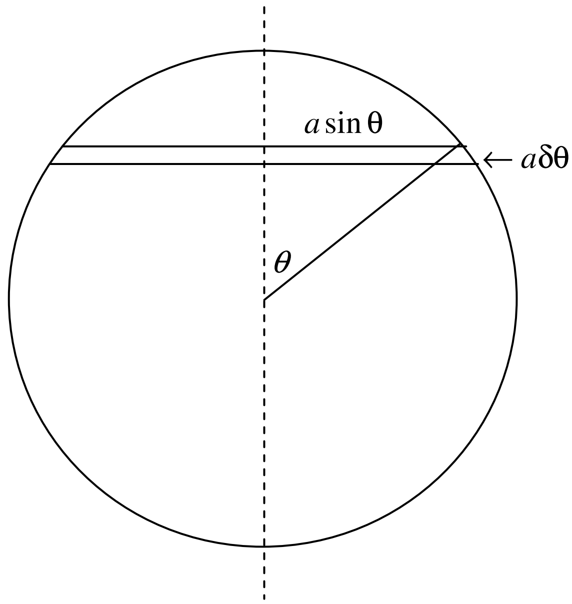
$\dfrac{3m}{20}\left(a^2 + 4h^2\right)$



$\dfrac{m}{20}\left(3a^2 + 2h^2\right)$

2.7 *Three-dimensional hollow figures. Spheres, cylinders, cones.*

Hollow spherical shell, mass m, radius a.



The area of the elemental zone is $2\pi a^2 \sin\theta\,\delta\theta$. Its mass is

$$m \times \frac{2\pi a^2 \sin\theta\delta\theta}{4\pi a^2} = \tfrac{1}{2} m \sin\theta\delta\theta.$$

Its moment of inertia is $\tfrac{1}{2} m \sin\theta\,\delta\theta \times a^2 \sin^2\theta = \tfrac{1}{2} m a^2 \sin^3\theta\,\delta\theta.$
The moment of inertia of the entire spherical shell is

$$\tfrac{1}{2} m a^2 \int_0^\pi \sin^3\theta d\theta = \tfrac{2}{3} m a^2.$$

This result can be used to calculate, by integration, the moment of inertia $\frac{2}{5}ma^2$ of a solid sphere. Or, if you start with $\frac{2}{5}ma^2$ for a solid sphere, you can differentiate to find the result $\frac{2}{3}ma^2$ for a hollow sphere. Write the moment of inertia for a solid sphere in terms of its density rather than its mass. Then add a layer *da* and calculate the increase *dI* in the moment of inertia. We can also use the moment of inertia for a hollow sphere ($\frac{2}{3}ma^2$) to calculate the moment of inertia of a nonuniform solid sphere in which the density varies as $\rho = \rho(r)$. For example, if $\rho = \rho_0\sqrt{1 - (r/a)^2}$, see if you can show that the mass of the sphere is $2.467\rho_0 a^3$ and that its moment of inertia is $\frac{1}{3}ma^2$. A much easier method will be found in Section 19.

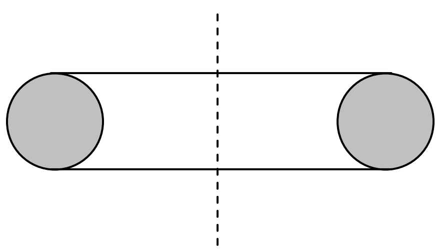
Using methods similar to that given for a solid cylinder, it is left as an exercise to show that the moment of inertia of an open hollow cylinder about an axis perpendicular to its length passing through its centre of mass is $m\left(\frac{1}{2}a^2 + \frac{1}{3}l^2\right)$, where a is the radius and $2l$ is the length.

The moment of inertia of a baseless hollow cone of mass *m,* base radius a, about the axis of the cone could be found by integration. However, those who have an understanding of the way in which the moment of inertia depends on the distribution of mass should readily see, without further ado, that the moment of inertia is $\frac{1}{2}ma^2$. (Look at the cone from above; it looks just like a disc, and indeed it has the same radial mass distribution as a uniform disc.)

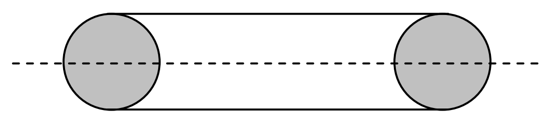
2.8 *Torus*

The rotational inertias of solid and hollow toruses (large radius a, small radius b) are given below for reference and without derivation. They can be derived by integral calculus, and their derivation is recommended as a challenge to the reader.

Solid torus:

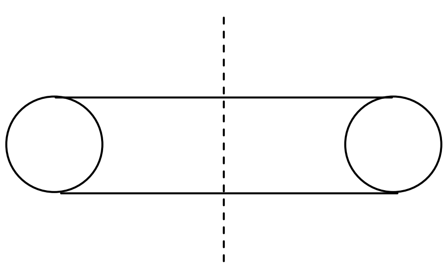


$\frac{1}{4}m\left(4a^2 + 3b^2\right)$

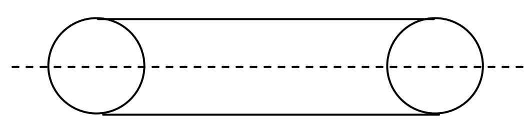


$\frac{1}{8}m\left(4a^2 + 5b^2\right)$

Hollow torus:

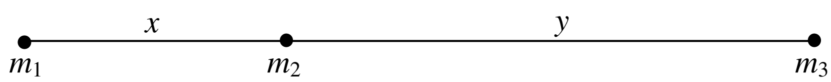


$\frac{1}{2}m\left(2a^2 + 3b^2\right)$



$\frac{1}{4}m\left(2a^2 + 5b^2\right)$

2.9 *Linear triatomic molecule*



Here is an interesting problem. It should be straightforward to calculate the rotational inertia of the above molecule with respect to an axis perpendicular to the molecule and passing through the centre of mass. In practice it is quite easy to *measure* the rotational inertia very precisely from the spacing between the lines in a molecular band in the infrared region of the spectrum. If you know the three masses (which you do if you know the atoms that make up the molecule) can you calculate the two interatomic spacings x and y ? That would require determining two unknown quantities, x and y, from a single measurement of the rotational inertia, I. Evidently that cannot be done; a second measurement is required. Can you suggest what might be done? We shall answer that shortly. In the meantime, it is an exercise to show that the rotational inertia is given by

$$ax^2 + 2hxy + by^2 + c = 0, \qquad 2.9.1$$

where

$$a = m_1(m_2 + m_3)/M \qquad 2.9.2$$

$$h = m_1 m_3 / M \qquad 2.9.3$$

$$b = m_3(m_1 + m_2)/M \qquad 2.9.4$$

$$M = m_1 + m_2 + m_3 \qquad 2.9.5$$

$$c = -I \qquad 2.9.6$$

For example, suppose the molecule is the linear molecule OCS, and the three masses are 16, 12 and 32 respectively, and, from infrared spectroscopy, it is determined that the moment of inertia is 20. (For this hypothetical illustrative example, I am not concerning myself with units). In that case, equation 2.9.1 becomes

$$11.7\dot{3}x^2 + 17.0\dot{6}xy + 14.9\dot{3}y^2 - 20 = 0. \qquad 2.9.7$$

We need another equation to solve for x and y. What can be done chemically is to prepare an isotopically-substituted molecule (isotopomer) such as ^{18}OCS, and measure *its* moment of inertia from its spectrum, making the probably very justified assumption that the interatomic distances are unaffected by the isotopic substitution. This results in a second equation:

$$a'x^2 + 2h'xy + b'y^2 + c' = 0. \qquad 2.9.8$$

Let's suppose that the new moment of inertia is $I' = 21$, and I leave it to the reader to work out the numerical values of a', h' and b' with the stern caution to retain all the decimal places on your calculator. That is, do not round off the numbers until the very end of the calculation.

You now have two equations, 2.9.1 and 2.9.8, to solve for x and y. These are two simultaneous quadratic equations, and it may be that some guidance in solving them would be helpful. I have three suggestions.

1. Treat equation 2.9.1 as a quadratic equation in x and solve it for x in terms of y. Then substitute this in equation 2.9.8. I expect you will *very* soon become bored with this method and will want to try something a little less tedious.

2. You have two equations of the form $S(x,y) = 0, \quad S'(x,y) = 0$. There are standard ways of solving these iteratively by an extension of the Newton-Raphson process. This is described, for example, in section 1.9 of Chapter 1 of my **Celestial Mechanics** notes, and this general method for two or more nonlinear equations should be known by anyone who expects to engage in much numerical calculation.

For this particular case, the detailed procedure would be as follows. This is an iterative method, and it is first necessary to make a guess at the solutions for x and y. The guesses need not be particularly good. That done, compute the following six quantities:

$$S = x(ax + 2hy) + by^2 + c$$

$$S' = x(a'x + 2h'y) + b'y^2 + c'$$

$$S_x = 2(ax + hy)$$

$$S_y = 2(hx + by)$$

$$S'_x = 2(a'x + h'y)$$

$$S'_y = 2(h'x + b'y)$$

Here the subscripts denote the partial derivatives. Now if

$$x(\text{true}) = x(\text{guess}) + \varepsilon$$

and

$$y(\text{true}) = y(\text{guess}) + \eta$$

the errors ε and η can be found from the solution of

$$S_x\varepsilon + S_y\eta + S = 0$$

and

$$S'_x\varepsilon + S'_y\eta + S' = 0$$

If we calculate

$$F = \frac{1}{S_yS'_x - S_xS'_y}$$

The solutions for the errors are

$$\varepsilon = F(S'_y S - S_y S')$$

$$\eta = F(S_x S' - S'_x S)$$

This will enable a better guess to be made, and the procedure can be repeated until the errors are as small as desired. Generally only a very few iterations are required. If this is not the case, a programming mistake is indicated.

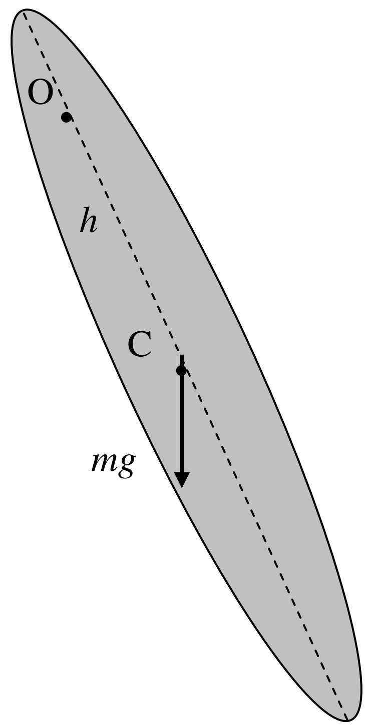
3. While method 2 can be used for any nonlinear simultaneous equations, in this particular case we have two simultaneous quadratic equations, and a little familiarity with conic sections provides a rather nice method.

   Thus, if $S = 0$ and $S' = 0$ are equations 2.9.1 and 2.9.8 respectively. Each of these equations represents a conic section, and they intersect at four points. We wish to find the point of intersection that lies in the all-positive quadrant - i.e. with x and y both positive. Since the two conic sections are very similar, in order to calculate where they intersect it is necessary to calculate with great accuracy. Therefore, do not round off the numbers until the very end of the calculation. Form the equation $c'S - cS' = 0$. This is also a quadratic equation representing a conic section passing through the four points. Furthermore, it has no constant term, and it therefore represents the two straight lines that pass through the four points. The equation can be factorized into two linear terms, $\alpha\beta = 0$, where $\alpha = 0$ and $\beta = 0$ are the two straight lines. Choose the one with positive slope and solve it with $S = 0$ or with $S' = 0$ (or with both, as a check against arithmetic mistakes) to find x and y. In this case, the solutions are $x = 0.2529$, $y = 1.000$.

2.10 *Pendulums*

In section 2.2, we discussed the physical meaning of the rotational inertia as being the ratio of the applied torque to the resulting angular acceleration. In linear motion, we are familiar with the equation $F = ma$. The corresponding equation when dealing with torques and angular acceleration is $\tau = I\ddot{\theta}$. We are also familiar with the equation of motion for a mass vibrating at the end of a spring of force constant k: $m\ddot{x} = -kx$. This is simple harmonic motion of period $2\pi\sqrt{m/k}$. The mechanics of the *torsion pendulum* is similar. The *torsion constant c* of a wire is the torque required to twist it through unit angle. If a mass is suspended from a torsion wire, and the wire is twisted through an angle θ, the restoring torque will be $c\theta$, and the equation of motion is $I\ddot{\theta} = -c\theta$, which is simple harmonic motion of period $2\pi\sqrt{I/c}$. The torsion constant of a wire of circular cross-section, by the way, is proportional to its shear modulus, the fourth power of its radius, and inversely as its length. The derivation of this takes a little trouble, but it can be verified by dimensional analysis. Thus a thick wire is very much harder to twist than a thin one. A wire of narrow rectangular cross-section, such as a strip or a ribbon has a relatively small torsion constant.

Now let's look not at a torsion pendulum, but at a pendulum swinging about an axis under gravity.



We suppose the pendulum, of mass m, is swinging about a point O, which is at a distance h from the centre of mass C. The rotational inertia about O is I. The line OC makes an angle θ with the vertical, so that the horizontal distance between O and C is $h \sin \theta$. The torque about O is $mgh \sin\theta$, so that the equation of motion is

$$I\ddot{\theta} = -mgh\sin\theta. \qquad 2.10.1$$

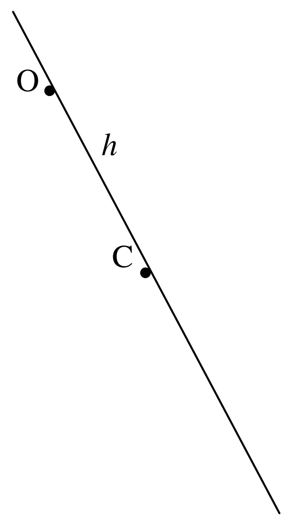
For small angles, this is

$$I\ddot{\theta} = -mgh\theta. \qquad 2.10.2$$

This is simple harmonic motion of period

$$P = 2\pi\sqrt{\frac{I}{mgh}}. \qquad 2.10.3$$

We'll look at two examples - a uniform rod, and an arc of a circle.
First, a uniform rod.



The centre of mass is C. The rotational inertia about C is $\frac{1}{3}ml^2$, so the rotational inertia about O is $I = \frac{1}{3}ml^2 + mh^2$. If we substitute this in equation 2.10.3, we find for the period of small oscillations

$$P = 2\pi\sqrt{\frac{l^2 + 3h^2}{3gh}}. \qquad 2.10.4$$

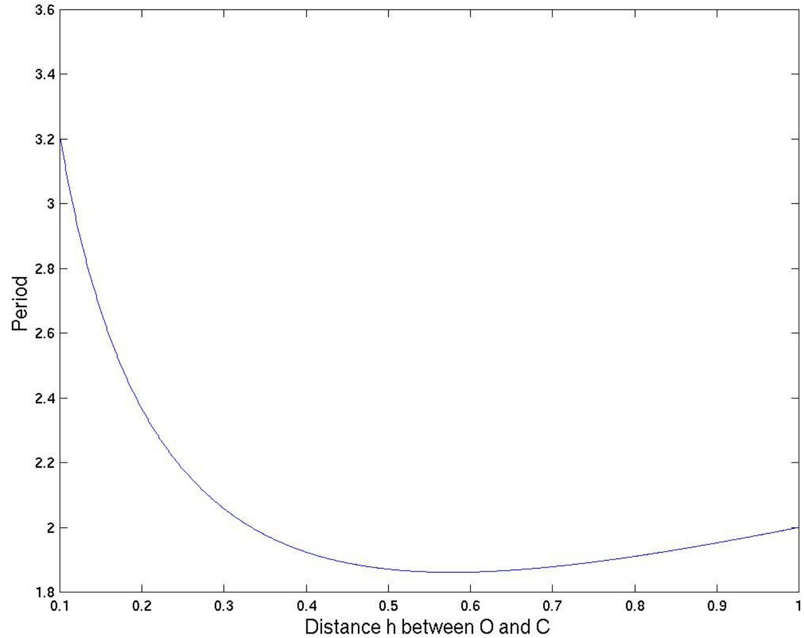
This can be written

$$P = 2\pi\sqrt{\frac{l}{3g}}.\sqrt{\frac{1 + 3(h/l)^2}{h/l}}, \qquad 2.10.5$$

or, if we write $\mathsf{P} = \frac{P}{2\pi\sqrt{\frac{l}{3g}}}$ and $\mathsf{h} = h/l$:

$$\mathsf{P} = \sqrt{\frac{1 + 3\mathsf{h}^2}{\mathsf{h}}}. \qquad 2.10.6$$

The figure shows a graph of P versus h.



Equation 2.10.6 can be written

$$P^2 = \frac{1}{h} + 3h \qquad 2.10.7$$

and, by differentiation of P^2 with respect to h, we find that the period is least when $h = 1/\sqrt{3}$. This least period is given by $P^2 = \sqrt{12}$, or $P = 1.861$.

Equation 2.10.7 can also be written

$$3h^2 - P^2 h + 1 = 0. \qquad 2.10.8$$

This quadratic equation shows that there are two positions of the support O that give rise to the same period of small oscillations. The period is least when the two solutions of equation 2.10.8 are equal, and by the theory of quadratic equations, then, the least period is given by $P^2 = \sqrt{12}$, as we also deduced by differentiation of equation 2.10.7, and this occurs when $h = 1/\sqrt{3}$.

For periods longer than this, there are two solutions for h. Let h_1 be the smaller of these, and let h_2 be the larger. By the theory of quadratic equations, we have

$$h_1 + h_2 = \tfrac{1}{3}P^2 \qquad 2.10.9$$

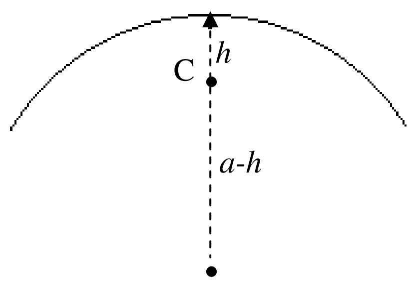
and $$h_1 h_2 = 1/3. \tag{2.10.10}$$

Let $H = h_2 - h_1$ be the distance between two points O that give the same period of oscillation. Then

$$H^2 = (h_2 - h_1)^2 = (h_2 + h_1)^2 - 4h_1 h_2 = \frac{P^4 - 12}{9}. \tag{2.10.11}$$

If we measure H for a given period P and recall the definition of P we see that this provides a method for determining g. Although this is a common undergraduate laboratory exercise, the graph shows that the minimum is very shallow and consequently H and hence g are very difficult to measure with any precision.

For another example, let us look at a wire bent into the arc of a circle of radius a oscillating in a vertical plane about its mid-point. In the figure, C is the centre of mass.



The rotational inertia about the centre of the circle is ma^2. By two applications of the parallel axes theorem, we see that the rotational inertia about the point of oscillation is $I = ma^2 - m(a-h)^2 + mh^2 = 2mah$. Thus, from equation 2.10.3 we find

$$P = 2\pi\sqrt{\frac{2a}{g}}, \tag{2.10.12}$$

and the period is independent of the length of the arc.

2.11. *Plane Laminas. Product moment. Translation of Axes (Parallel Axes Theorem).*

We consider a set of point masses distributed in a plane, or a plane lamina. We have hitherto met three second moments of inertia:

$$A = \sum m_i y_i^2, \tag{2.11.1}$$

$$B = \sum m_i x_i^2, \tag{2.11.2}$$

$$C = \sum m_i \left(x_i^2 + y_i^2 \right). \qquad 2.11.3$$

These are respectively the moments of inertia about the *x*- and *y*-axes (assumed to be in the plane of the masses or the lamina) and the *z*-axis (normal to the plane). Clearly, $C = A + B$, which is the perpendicular axes theorem for a plane lamina.

We now introduce another quantity, H, called the *product moment of inertia* with respect to the *x*- and *y*-axes, defined by

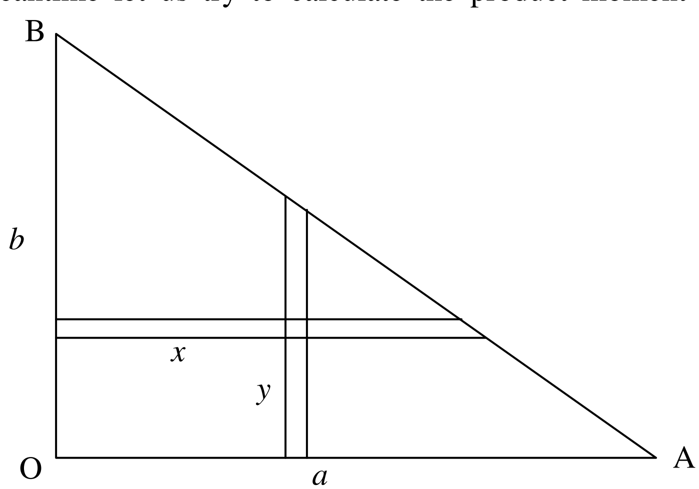
$$H = \sum m_i x_i y_i . \qquad 2.11.4$$

We'll need sometime to ask ourselves whether this has any particular physical significance, or whether it is merely something to calculate for the sake of passing the time of day. In the meantime, the reader should recall the parallel axes theorems (Section 2.5) and, using arguments similar to those given in that section, should derive

$$H = H_C + M\bar{x}\bar{y}. \qquad 2.11.5$$

It may also be noted that equation 2.11.4 does not contain any squared terms and therefore the product moment of inertia, depending on the distribution of masses, is just as likely to be a negative quantity as a positive one.

We shall defer discussing the physical significance, if any, of the product moment until section 12. In the meantime let us try to calculate the product moment for a plane right triangular lamina:



The area of the triangle is $\frac{1}{2}ab$ and so the mass of the element $\delta x \delta y$ is $\frac{2M\delta x\delta y}{ab}$, where M is the mass of the complete triangle. The product moment of the element with respect to the sides OA, OB is $\frac{2Mxy\,\delta x\,\delta y}{ab}$ and so the product moment of the entire triangle is $\frac{2M}{ab}\iint xydxdy$. We have to consider carefully the limits of integration. We'll integrate first with respect to x ; that is to say we integrate along the horizontal (y constant) strip from the side OB to the side AB. That is to say we integrate $x\delta x$ from where $x = 0$ to where $x = a\left(1 - \frac{y}{b}\right)$. The product moment is therefore

$$\frac{2M}{ab}\int y.\tfrac{1}{2}a^2\left(1 - \tfrac{y}{b}\right)^2 dy.$$

We now have to add up all the horizontal strips from the side OA, where $y = 0$, to B, where $y = b$. Thus

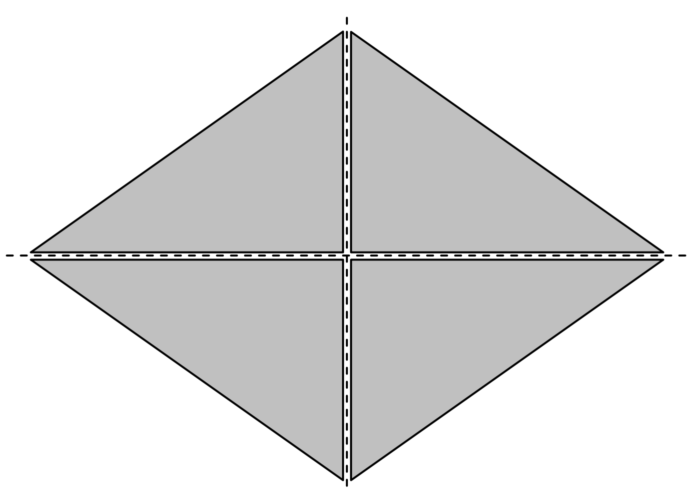
$$H = \frac{Ma}{b}\int_0^b y\left(1 - \tfrac{y}{b}\right)^2 dy\,,$$

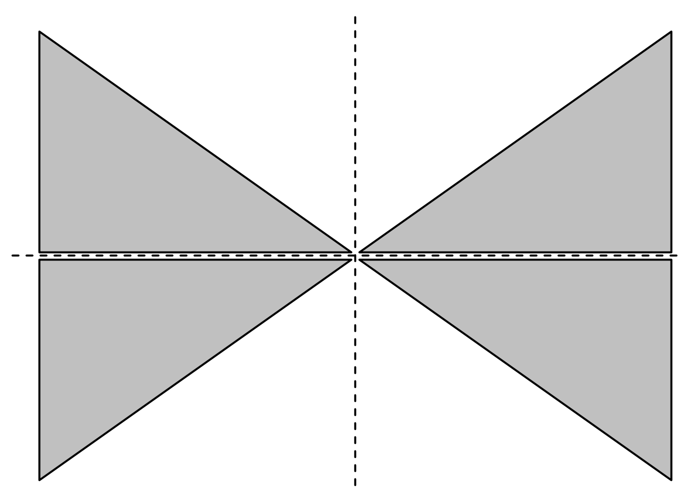
which, after some algebra, comes to $H = \frac{1}{12}Mab.$

The coordinates of the centre of mass with respect to the sides OA, OB are $\left(\frac{1}{3}a,\ \frac{1}{3}b\right)$, so that, from equation 2.11.5, we find that the product moment with respect to axes parallel to OA, OB and passing through the centre of mass is $-\frac{1}{36}Mab.$

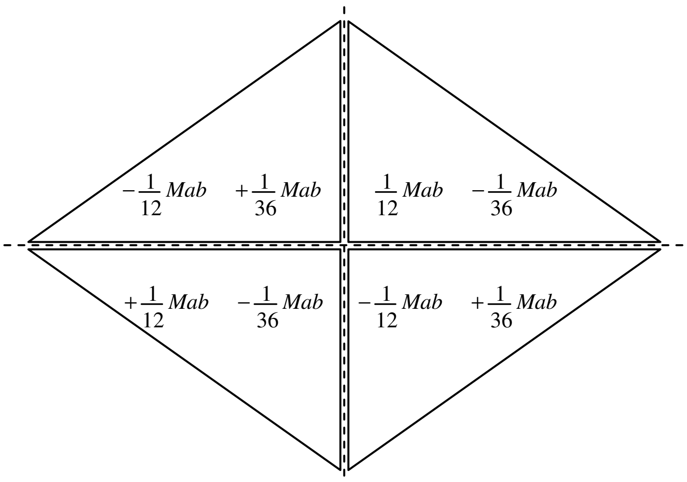
Exercise:

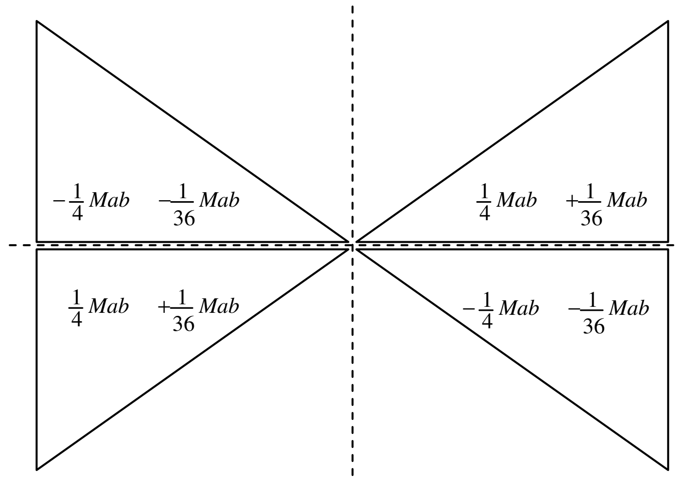
Calculate the product moments of the following eight laminas, each of mass M, with respect to horizontal and vertical axes through the origin, and with respect to horizontal and vertical axes through the centroid of each. (We have just done the first of these, above.) The horizontal base of each is of length a, and the height of each is b. You are going to have to take great care with the *signs*, and with the limits of integration. If you get an answer right except for the *sign*, then you have got the answer *wrong*.



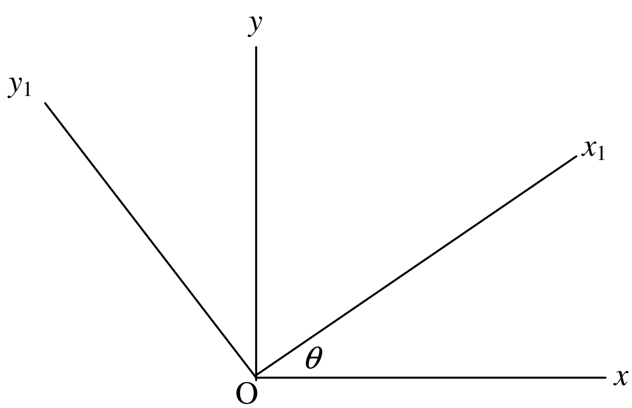


I make the answers as follows.





2.12 *Rotation of Axes.*



We start by recalling a result from elementary geometry. Consider two sets of axes Oxy and Ox_1y_1, the latter being inclined at an angle θ to the former. Any point in the plane can be described by the coordinates (x , y) or by (x_1 , y_1). These coordinates are related by a rotation matrix:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad 2.12.1$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \qquad 2.12.2$$

The rotation matrix is orthogonal; one of the several properties of an orthogonal matrix is that its reciprocal is its transpose.

Now let us apply this to the moments of inertia of a plane lamina. Let us suppose that the axes are in the plane of the lamina and that O is the centre of mass of the lamina. *A, B* and *H* are the moments of inertia with respect to the axes Oxy, and A_1 , B_1 and H_1 are the moments of inertia with respect to Ox_1y_1. Strictly speaking a lamina implies a continuous distribution of matter in a plane, but, since matter, we are told, is composed of discrete atoms, there is little difficulty in justifying treating a lamina as though it we a distribution of point masses in the plane. In any case the results that follow are valid either for a collection of point masses in a plane or for a genuine continuous lamina.

We have, by definition:

$$A_1 = \sum m y_1^2 \qquad 2.12.3$$

$$B_1 = \sum m x_1^2 \qquad 2.12.4$$

$$H_1 = \sum m x_1 y_1 \qquad 2.12.5$$

Now let us apply equation 2.12.1 to equation 2.12.3:

$$A_1 = \sum m(-x\sin\theta + y\cos\theta)^2 = \sin^2\theta \sum mx^2 - 2\sin\theta\cos\theta \sum mxy + \cos^2\theta \sum my^2.$$

That is to say (writing the third term first, and the first term last)

$$A_1 = A\cos^2\theta - 2H\sin\theta\cos\theta + B\sin^2\theta. \qquad 2.12.6$$

In a similar fashion, we obtain for the other two moments

$$B_1 = A\sin^2\theta + 2H\sin\theta\cos\theta + B\cos^2\theta \qquad 2.12.7$$

and

$$H_1 = A\sin\theta\cos\theta + H(\cos^2\theta - \sin^2\theta) - B\sin\theta\cos\theta. \qquad 2.12.8$$

It is usually more convenient to make use of trigonometric identities to write these as

$$A_1 = \tfrac{1}{2}(B+A) - \tfrac{1}{2}(B-A)\cos 2\theta - H\sin 2\theta, \qquad 2.12.9$$

$$B_1 = \tfrac{1}{2}(B+A) + \tfrac{1}{2}(B-A)\cos 2\theta + H\sin 2\theta, \qquad 2.12.10$$

$$H_1 = H\cos 2\theta - \tfrac{1}{2}(B-A)\sin 2\theta. \qquad 2.12.11$$

These equations enable us to calculate the moments of inertia with respect to the axes Ox_1y_1 if we know the moments with respect to the axes Oxy.

Further, a matter of importance, we see, from equation 2.12.11, that if

$$\tan 2\theta = \frac{2H}{B - A}, \qquad 2.12.12$$

the product moment H_1 *with respect to the axes* Oxy *is zero.* This gives some physical meaning to the product moment, namely: If we can find some axes (which we *can*, by means of equation 2.12.12) with respect to which the product moment is zero, these axes are called the *principal axes* of the lamina, and the moments of inertia with respect to the principal axes are called the *principal moments of inertia*. I shall use the symbols A_0 and B_0 for the principal moments of inertia, and I shall adopt the convention that $A_0 \le B_0$.

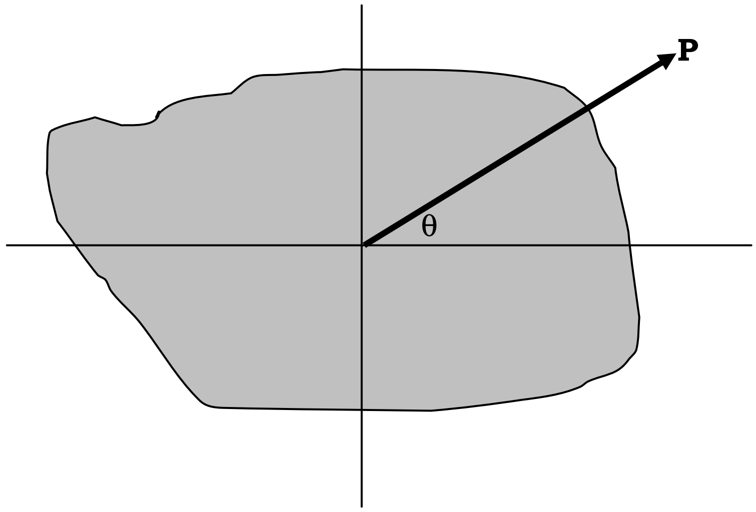
Example: Consider three point masses at the coordinates given below:

Mass	Coordinates
5	(1 , 1)
3	(4 , 2)
2	(3 , 4)

The moments of inertia are $A = 49$, $B = 71$, $C = 53$. The coordinates of the centre of mass are (2.3 , 1.9). If we use the parallel axes theorem, we can find the moments of inertia with respect to axes parallel to the original ones but with origin at the centre of mass. With respect to these axes we find $A = 12.9$, $B = 18.1$, $H = +9.3$. The principal axes are therefore inclined at angles θ to the x-axis given (equation 2.13.12) by $\tan 2\theta = 3.57669$; That is $\theta = 37^\circ\ 11'$ and $127^\circ\ 11'$. On using equation 2.12.9 or 10 with these two angles, together with the convention that $A_0 \le B_0$, we obtain for the principal moments of inertia $A_0 = 5.84$ and $B_0 = 25.16$.

Example. Consider the right-angled triangular lamina of section 11. The moments of inertia with respect to axes passing through the centre of mass and parallel to the orthogonal sides of the triangle are $A = \frac{1}{18}Mb^2$, $B = \frac{1}{18}Ma^2$, $H = -\frac{1}{36}Mab$. The angles that the principal axes make with the a - side are given by $\tan 2\theta = \dfrac{ab}{b^2 - a^2}\cdot$ The interested reader will be able to work out expressions, in terms of M, a, b, for the principal moments.

2.13 *Momental Ellipse*



Consider a plane lamina such that its radius of gyration about some axis through the centre of mass is k. Let $\mathbf{P}$ be a vector in the direction of that axis, originating at the centre of mass, given by

$$\mathbf{P} = \frac{a^2}{k}\hat{\mathbf{r}} \qquad 2.13.1$$

Here $\hat{\mathbf{r}}$ is a unit vector in the direction of interest; k is the radius of gyration, and a is an arbitrary length introduced so that the dimensions of $\mathbf{P}$ are those of length, and the length of the vector $\mathbf{P}$ is inversely proportional to the radius of gyration. The moment of inertia is $Mk^2 = Ma^4 / P^2$. That is to say

$$\frac{Ma^4}{P^2} = A\cos^2\theta - 2H\sin\theta\cos\theta + B\sin^2\theta, \qquad 2.13.2$$

where A, H and B are the moments with respect to the x- and y-axes. Let $(x\ ,\ y)$ be the coordinates of the tip of the vector $\mathbf{P}$, so that $x = P\cos\theta$ and $y = P\sin\theta$. Then

$$Ma^4 = Ax^2 - 2Hxy + By^2. \qquad 2.13.3$$

Thus, no matter what the shape of the lamina, however irregular and asymmetric, the tip of the vector $\mathbf{P}$ traces out an ellipse, whose axes are inclined at angles $\frac{1}{2}\tan^{-1}\left(\frac{2H}{B-A}\right)$ to the x-axis. This is the *momental ellipse,* and the axes of the momental ellipse are the principal axes of the lamina.

Example. Consider a regular n-gon. By symmetry the moment of inertia is the same about any two axes in the plane inclined at $2\pi/n$ to each other. This is possible only if the momental ellipse is a circle. It follows that the moment of inertia of a uniform polygonal plane lamina is the same about any axis in its plane and passing through its centroid.

Exercise. Show that the moment of inertia of a uniform plane n-gon of side $2a$ about any axis in its plane and passing through its centroid is $\frac{1}{12}ma^2\left(1+3\cot^2(\pi/n)\right)$. What is this for a square? For an equilateral triangle?

2.14. *Eigenvectors and eigenvalues.*

In sections 11-13, we have been considering some aspects of the moments of inertia of plane laminas, and we have discussed such matters as rotation of axes, and such concepts as product moments of inertia, principal axes, principal moments of inertia and the momental ellipse. We next need to develop the same concepts with respect to three-dimensional solid bodies. In doing so, we shall need to make use of the algebraic concepts of eigenvectors and eigenvalues. If you are already familiar with such matters, you may want to skip this section and move on to the next. If the ideas of eigenvalues and eigenvectors are new to you, or if you are a bit rusty with them, this section may be helpful. I do assume that the reader is at least familiar with the elementary rules of matrix multiplication.

Consider what happens when you multiply a vector, for example the vector $\begin{pmatrix}0\\1\end{pmatrix}$, by a square matrix, for example the matrix $\begin{pmatrix}4 & -1\\2 & 1\end{pmatrix}$. We obtain:

$$\begin{pmatrix}4 & -1\\2 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}.$$

The result of the operation is another vector that is in quite a different direction from the original one.

However, now let us multiply the vector $\begin{pmatrix}1\\1\end{pmatrix}$ by the same matrix. The result is $\begin{pmatrix}3\\3\end{pmatrix}$. The result of the multiplication is merely to multiply the vector by 3 without changing its direction. The vector $\begin{pmatrix}1\\1\end{pmatrix}$ is a very special one, and it is called an *eigenvector* of the matrix, and the multiplier 3 is called the corresponding *eigenvalue*. "Eigen" is German for "own" in the sense of "my own book". There is one other eigenvector of the matrix; it is the vector $\begin{pmatrix}1\\2\end{pmatrix}$. Try it; you should find that the corresponding eigenvalue is 2.

In short, given a square matrix **A**, if you can find a vector **a** such that **Aa** = λ**a**, where λ is merely a scalar multiplier that does not change the direction of the vector **a**, then **a** is an eigenvector and λ is the corresponding eigenvalue.

In the above, I told you what the two eigenvectors were, and you were able to verify that they were indeed eigenvectors and you were able to find their eigenvalues by straightforward arithmetic. But, what if I hadn't told you the eigenvectors? How would you find them?

Let **A**=$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and let **x**=$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigenvector with corresponding eigenvalue λ. Then we must have

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}.$$

That is,

$$(A_{11} - \lambda)x_1 + A_{12}x_2 = 0$$

and

$$A_{21}x_1 + (A_{22} - \lambda)x_2 = 0.$$

These two equations are consistent only if the determinant of the coefficients is zero. That is,

$$\begin{vmatrix} A_{11} - \lambda & A_{12} \\ A_{21} & A_{22} - \lambda \end{vmatrix} = 0.$$

This equation is a quadratic equation in λ, known as the *characteristic equation*, and its two roots, the *characteristic* or *latent roots* are the eigenvalues of the matrix. Once the eigenvalues are found the ratio of x_1 to x_2 is easily found, and hence the eigenvectors.

Similarly, if **A** is a 3 × 3 matrix, the characteristic equation is

$$\begin{vmatrix} A_{11} - \lambda & A_{12} & A_{13} \\ A_{21} & A_{22} - \lambda & A_{23} \\ A_{31} & A_{32} & A_{33} - \lambda \end{vmatrix} = 0.$$

This is a cubic equation in λ, the three roots being the eigenvalues. For each eigenvalue, the ratio $x_1 : x_2 : x_3$ can easily be found and hence the eigenvectors. The characteristic equation is a cubic equation, and is best solved numerically, not by algebraic formula. The cubic equation can be written in the form

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

and the solutions can be checked from the following results from the theory of equations:

$$\lambda_1 + \lambda_2 + \lambda_3 = -a_2,$$

$$\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 = a_1,$$

$$\lambda_1\lambda_2\lambda_3 = -a_0.$$

2.15. *Solid body.*

The moments of inertia of a collection of point masses distributed in three-dimensional space (or of a solid three-dimensional body, which, after all, is a collection of point masses (atoms)) with respect to axes O*xyz* are:

$$A = \sum m(y^2+z^2) \qquad F = \sum myz$$

$$B = \sum m(z^2+x^2) \qquad G = \sum mzx$$

$$C = \sum m(x^2+y^2) \qquad H = \sum mxy$$

Suppose that $A, \ B, \ C, \ F, \ G, \ H,$ are the moments and products of inertia with respect to axes whose origin is at the centre of mass. The *parallel axes theorems* (which the reader should prove) are as follows: Let P be some point not at the centre of mass, such that the coordinates of the centre of mass with respect to axes parallel to the axes O*xyz* but with origin at P are $(\overline{x},\overline{y},\overline{z})$. Then the moments and products of inertia with respect to the axes through P are

$$A + M(\overline{y}^2 + \overline{z}^2) \qquad F + M\overline{yz}$$
$$B + M(\overline{z}^2 + \overline{x}^2) \qquad G + M\overline{zx}$$
$$C + M(\overline{x}^2 + \overline{y}^2) \qquad H + M\overline{xy}$$

where M is the total mass.

Unless stated otherwise, in what follows we shall suppose that the moments and products of inertia under discussion are referred to a set of axes with the centre of mass as origin.

2.16 *Rotation of axes - three dimensions.*

Let Oxyz be one set of mutually orthogonal axes, and let O$x_1y_1z_1$ be another set of axes inclined to the first. The coordinates (x_1 , y_1 , z_1) of a point with respect to the second basis set are related to the coordinates (x, y, z) with respect to the first by

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \qquad 2.16.1$$

Here the c_{ij} are the cosines of the angles between the axes of one basis set with respect to the axes of the other. For example, c_{12} is the cosine of the angle between Ox_1 and Oy. c_{23} is the cosine of the angles between Oy_1 and Oz.

Some readers may know how to express these cosines in terms of complicated expressions involving the *Eulerian angles.* While these are important, they are not essential for following the present development, so we shall not make use of the Eulerian angles just here.

The matrix of direction cosines is *orthogonal*. Among the several properties of an orthogonal matrix is the fact that its reciprocal (inverse) is equal to its transpose - i.e. the reciprocal of an orthogonal matrix is found merely my interchanging the rows and columns. This enables us easily to find (x , y , z) in terms of (x_1 , y_1 , z_1).

A number of other properties of an orthogonal matrix are useful in detecting, locating and even correcting arithmetic mistakes in computing the elements. These properties are

1. The sum of the squares of the elements in any row or column is unity. This merely expresses the fact that the magnitude of a unit vector along any of the six axes is indeed unity.

2. The sum of the products of corresponding elements of any two rows or of any two columns is zero. This merely expresses the fact that the scalar product of any two orthogonal vectors is zero. It will be noted that checking for property 1 will not detect any mistakes in sign of the elements, whereas checking for property 2 will do so.

3. Every element is equal to ± its own cofactor. This expresses the fact that the cross product of two unit orthogonal vectors is equal to the third.

4. The determinant of the matrix is ± 1. If the sign is negative, it means that the chiralities (handedness) of the two basis sets of axes are opposite; i.e. one of them is a right-handed set and the other is a left-handed set. It is usually convenient to choose both sets as right-handed.

If it is possible to find a set of axes with respect to which the product moments F, G and H are all zero, these axes are called the principal axes *of the body, and the moments of inertia with respect to these axes are the* principal moments of inertia*,* for which we shall use the notation A_0 , B_0 , C_0, with the convention $A_0 \le B_0 \le C_0$. We shall see shortly that it is indeed possible, and we shall show how to do it. A vector whose length is inversely proportional to the radius of gyration traces out in space an ellipsoid, known as the *momental ellipsoid.*

In the study of solid body rotation (whether by astronomers studying the rotation of asteroids or by chemists studying the rotation of molecules) bodies are classified as follows.

1. $A_0 \ne B_0 \ne C_0$ The ellipsoid is a triaxial ellipsoid, and the body is an *asymmetric top.*
2. $A_0 < B_0 = C_0$ The ellipsoid is a prolate spheroid and the body is a *prolate symmetric top.*
3. $A_0 = B_0 < C_0$ The ellipsoid is an oblate spheroid and the body is an *oblate symmetric top.*
4. $A_0 = B_0 = C_0$ The ellipsoid is a sphere and the body is a *spherical top.*
5. One moment is zero. The ellipsoid is an infinite elliptical cylinder, and the body is a *linear top*.

Example. We know from section 2.5 that the moment of inertia of a plane square lamina of side $2a$ about an axis through its centroid and perpendicular to its area is $\frac{2}{3}ma^2$, and it will hence be obvious that the moment of inertia of a uniform solid cube of side $2a$ about an axis passing through the mid-points of opposite sides is also $\frac{2}{3}ma^2$. It will clearly be the same about an axis passing through the mid-points of *any* pairs of opposite sides. Therefore the cube is a spherical top and the momental ellipsoid is a sphere. Therefore the moment of inertia of a uniform solid cube about *any* axis through its centre (including, for example, a diagonal) is also $\frac{2}{3}ma^2$.

Example. What is the ratio of the length to the diameter of a uniform solid cylinder such that it is a spherical top? [Answer: I make it $\sqrt{3}/2 = 0.866$.]

Let us note in passing that

$$A + B + C = 2\sum m\left(x^2 + y^2 + z^2\right) = 2\sum mr^2, \qquad 2.16.2$$

which is independent of the orientation of the basis axes In other words, regardless of how *A*, *B* and *C* may depend on the orientation of the axes with respect to the body, the sum $A + B + C$ is invariant under a rotation of axes.

We shall deal with the determination of the principal axes in section 2.18 - but don't skip section 2.17.

2.17 *Solid Body Rotation. The Inertia Tensor.*

It is intended that this chapter should be limited to the calculation of the moments of inertia of bodies of various shapes, and not with the huge subject of the rotational dynamics of solid bodies, which requires a chapter on its own. In this section I mention merely for interest two small topics involving the principal axes, and a third topic in a bit more detail as necessary before proceeding to section 2.18.

Everyone knows that the relation between translational kinetic energy and linear momentum is $E = p^2/(2m)$. Similarly rotational kinetic energy is related to angular momentum L by $E = L^2/(2I)$, where I is the moment of inertia. If an isolated body (such as an asteroid) is rotating about a non-principal axis, it will be subject to internal stresses. If the body is *nonrigid* this will result in distortions (strains) which may cause the body to vibrate. If in addition the body is *inelastic* the vibrations will rapidly die out (if the damping is greater than critical damping, indeed, the body will not even vibrate). Energy that was originally rotational kinetic energy will be converted to heat (which will be radiated away.) The body loses rotational kinetic energy. In the absence of external torques, however, L remains constant. Therefore, while E diminishes, I increases. The body adjusts its rotation until it is rotating around its axis of maximum moment of inertia, at which time there are no further stresses, and the situation remains stable.

In general the rotational motion of a solid body whose momental ellipse is triaxial is quite complicated and chaotic, with the body tumbling over and over in apparently random fashion. However, if the body is nonrigid and inelastic (as all real bodies are in practice), it will eventually end up rotating about its axis of maximum moment of inertia. The time taken for a body, initially tumbling chaotically over and over, until it reaches its final blissful state of rotation about its axis of maximum moment of inertia, depends on how fast it is rotating. For most irregular small asteroids the time taken is comparable to or longer than the age of formation of the solar system, so that it is not surprising to find some asteroids with non-principal axis (NPA) rotation. However, a few rapidly-rotating NPA asteroids have been discovered, and, for rapid rotators, one would expect PA rotation to have been reached a long time ago. It is thought that something (such as a collision) must have happened to these rapidly-rotating NPA asteroids relatively recently in the history of the solar system.

Another interesting topic is that of the *stability* of a rigid rotator that is rotating about a principal axis, against small perturbations from its rotational state. Although I do not prove it here (the proof can be done either mathematically, or by a qualitative argument) rotation about either of the axes of maximum or of minimum moment of inertia is stable, whereas rotation about the intermediate axis is unstable. The reader can observe this for him- or herself. Find anything that is triaxial - such as a small block of wood shaped as a rectangular parallelepiped with unequal sides. Identify the axes of greatest, least and intermediate moment of inertia. Toss the body up in the air at the same time setting it rotating about one or the other of these axes, and you will be able to see for yourself that the rotation is stable in two cases but unstable in the third.

I now deal with a third topic in rather more detail, namely the relation between angular momentum $\mathbf{L}$ and angular velocity $\boldsymbol{\omega}$. The reader will be familiar from elementary (and two-dimensional) mechanics with the relation $L = I\omega$. What we are going to find in the three-dimensional solid-body case is that the relation is $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$. Here $\mathbf{L}$ and $\boldsymbol{\omega}$ are, of course, vectors, *but they are not necessarily parallel to each other.* They are parallel only if the body is rotating about a principal axis of rotation. The quantity $\mathbf{I}$ is a tensor known as the *inertia tensor.* Readers will be familiar with the equation $\mathbf{F} = m\mathbf{a}$. Here the two vectors are in the same direction, and m is a scalar quantity that does not change the direction of the vector that it multiplies. A tensor usually (unless its matrix representation is *diagonal*) changes the direction as well as the magnitude of the vector that it multiplies. The reader might like to think of other examples of tensors in physics. There are several. One that comes to mind is the permittivity of an anisotropic crystal; in the equation $\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$, $\mathbf{D}$ and $\mathbf{E}$ are not parallel unless they are both directed along one of the crystallographic axes.

If there are no external torques acting on a body, $\mathbf{L}$ is constant in both magnitude and direction. The instantaneous angular velocity vector, however, is not fixed either in space or with respect to the body - unless the body is rotating about a principal axis and the inertia tensor is diagonal.

So much for a preview and a qualitative description. Now down to work.

I am going to have to assume familiarity with the equation for the components of the cross product of two vectors:

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\hat{\mathbf{x}} + (A_z B_x - A_x B_z)\hat{\mathbf{y}} + (A_x B_y - A_y B_x)\hat{\mathbf{z}}. \qquad 2.17.1$$

I am also going to assume that the reader knows that the angular momentum of a particle of mass m at position vector $\mathbf{r}$ (components (x,y,z)) and moving with velocity $\mathbf{v}$ (components $(\dot{x},\dot{y},\dot{z})$) is $m\mathbf{r}\times\mathbf{v}$. For a collection of particles, (or an extended solid body, which, I'm told, consists of a collection of particles called atoms), the angular momentum is

$$\mathbf{L} = \sum m\mathbf{r}\times\mathbf{v}$$

$$= \sum\left[m(y\dot{z} - z\dot{y})\hat{\mathbf{x}} + m(z\dot{x} - x\dot{z})\hat{\mathbf{y}} + m(x\dot{y} - y\dot{x})\hat{\mathbf{z}}\right]$$

I also assume that the relation between linear velocity $\mathbf{v}$ $(\dot{x},\dot{y},\dot{z})$ and angular velocity $\boldsymbol{\omega}$ $(\omega_x,\omega_y,\omega_z)$ is understood to be $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, so that, for example, $\dot{z} = \omega_x y - \omega_y x$. Then

$$\mathbf{L} = \sum\left[m\big(y(\omega_x y - \omega_y x) - z(\omega_z x - \omega_x z)\big)\hat{\mathbf{x}} + (\text{etc.})\hat{\mathbf{y}} + (\text{etc.})\hat{\mathbf{z}}\right]$$

$$= \left(\omega_x \sum my^2 - \omega_y \sum mxy - \omega_z \sum mzx + \omega_x \sum mz^2\right)\hat{\mathbf{x}} + \text{etc.}$$

$$= (A\omega_x - H\omega_y - G\omega_z)\hat{\mathbf{x}} + (\quad)\hat{\mathbf{y}} + (\quad)\hat{\mathbf{z}}.$$

Finally, we obtain

$$\mathbf{L} = \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} A & -H & -G \\ -H & B & -F \\ -G & -F & C \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \qquad 2.17.2$$

This is the equation $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ referred to above. The inertia tensor is sometimes written in the form

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix},$$

so that, for example, $I_{xy} = -H$. It is a symmetric matrix (but it is not an orthogonal matrix).

2.18. *Determination of the Principal Axes.*

We now need to address ourselves to the determination of the principal axes. Unlike the two-dimensional case, we do not have a nice, simple explicit expression similar to equation 2.12.12 to calculate the orientations of the principal axes. The determination is best done through a numerical example.

Consider four masses whose positions and coordinates are as follows:

M	x	y	z
1	3	1	4
2	1	5	9
3	2	6	5
4	3	5	9

Relative to the first particle, the coordinates are

1	0	0	0
2	−2	4	5
3	−1	5	1
4	0	4	5

From this, it is easily found that the coordinates of the centre of mass relative to the first particle are (−0.7 , 3.9 , 3.3), and the moments of inertia with respect to axes through the first particle are

$$
\begin{aligned}
A &= 324\\
B &= 164\\
C &= 182\\
F &= 135\\
G &= -23\\
H &= -31
\end{aligned}
$$

From the parallel axes theorems we can find the moments of inertia with respect to axes passing through the centre of mass:

$$
\begin{aligned}
A &= 63.0\\
B &= 50.2\\
C &= 25.0\\
F &= \ \ 6.3\\
G &= \ \ 0.1\\
H &= -3.7
\end{aligned}
$$

The inertia tensor is therefore

$$
\begin{pmatrix} 63.0 & 3.7 & -0.1 \\ 3.7 & 50.2 & -6.3 \\ -0.1 & -6.3 & 25.0 \end{pmatrix}
$$

We understand from what has been written previously that if **ω**, the instantaneous angular velocity vector, is along any of the principal axes, then **Iω** will be in the same direction as **ω**. In other words, if (l,m,n) are the direction cosines of a principal axis, then

$$
\begin{pmatrix} A & -H & -G \\ -H & B & -F \\ -G & -F & C \end{pmatrix}\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \lambda\begin{pmatrix} l \\ m \\ n \end{pmatrix},
$$

where λ is a scalar quantity. In other words, a vector with components *l, m, n* (direction cosines of a principal axis) is an eigenvector of the inertia tensor, and λ is the corresponding principal moment of inertia. There will be three eigenvectors (at right angles to each other) and three corresponding eigenvalues, which we'll initially call λ_1, λ_2, λ_3, though, as soon as we know which is the largest and which the smallest, we'll call A_0, B_0, C_0, according to our convention $A_0 \le B_0 \le C_0$.

The characteristic equation is

$$\begin{vmatrix} A-\lambda & -H & -G \\ -H & B-\lambda & -F \\ -G & -F & C-\lambda \end{vmatrix} = 0.$$

In this case, this results in the cubic equation

$$a_0 + a_1\lambda + a_2\lambda^2 - \lambda^3 = 0,$$

where

$$\begin{aligned} a_0 &= 76226.44 \\ a_1 &= -5939.21 \\ a_2 &= \quad 138.20 \end{aligned}$$

The three solutions for λ, which we shall call A_0 , B_0 , C_0 in order of increasing size are

$$\begin{aligned} A_0 &= 23.498\,256 \\ B_0 &= 50.627\,521 \\ C_0 &= 64.074\,223 \end{aligned}$$

and these are the principal moments of inertia. From the theory of equations, we note that the sum of the roots is exactly equal to a_2, and we also note that it is equal to $A + B + C$, consistent with what we wrote in section 2.16. (See equation 2.16.2) The sum of the diagonal elements of a matrix is known as the *trace* of the matrix. Mathematically we say that "the trace of a symmetric matrix is invariant under an orthogonal transformation".

Two other relations from the theory of equations may be used as a check on the correctness of the arithmetic. The product of the solutions equals a_0 , which is also equal to the determinant of the inertia tensor, and the sum of the products taken two at a time equals $-a_1$.

We have now found the magnitudes of the principal moments of inertia; we have yet to find the direction cosines of the three principal axes. Let's start with the axis of least moment of inertia, for which the moment of inertia is $A_0 = 23.498\ 256$. Let the direction cosines of this axis be (l_1, m_1, n_1). Since this is an eigenvector with eigenvalue 23.498 256 we must have

$$\begin{pmatrix} 63.0 & 3.7 & -0.1 \\ 3.7 & 50.2 & -6.3 \\ -0.1 & -6.3 & 25.0 \end{pmatrix} \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} = 23.498256 \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}$$

These are three linear equations in l_1 m_1, n_1, with no constant term. Because of the lack of a constant term, the theory of equations tells us that the third equation, if it is consistent with the other two, must be a linear combination of the first two. We have, in effect, only two independent equations, and we are going to need a third, independent equation if we are to solve for the three direction cosines. If we let $l' = l_1/n_1$ and $m' = m_1/n_1$, then the first two equations become

$$39.501744l' + 3.7m' - 0.1 = 0$$
$$3.7l' + 26.701744m' - 6.3 = 0.$$

The solutions are

$$l' = -0.019825485$$
$$m' = +0.238686617.$$

The correctness of the arithmetic can and should be checked by verifying that these solutions also satisfy the third equation.

The additional equation that we need is provided by Pythagoras's theorem, which gives for the relation between three direction cosines

$$l_1^2 + m_1^2 + n_1^2 = 1,$$

or $$n_1^2 = \frac{1}{l'^2 + m'^2 + 1},$$

whence $$n_1 = \pm 0.972495608.$$

Thus we have, for the direction cosines of the axis corresponding to the moment of inertia A_0,

$$l_1 = \mp 0.019\,280\,197$$
$$m_1 = \pm 0.232\,121\,881$$
$$n_1 = \pm 0.972\,495\,608$$

(Check that $l_1^2 + m_1^2 + n_1^2 = 1.$)

It does not matter which sign you choose - after all, the principal axis goes both ways.

Similar calculations for B_0 yield

$$l_2 = \pm 0.280\,652\,440$$
$$m_2 = \mp 0.932\,312\,706$$
$$n_2 = \pm 0.228\,094\,774$$

and for C_0

$$l_3 = \pm\ 0.959\ 615\ 796$$
$$m_3 = \pm\ 0.277\ 330\ 987$$
$$n_3 = \mp\ 0.047\ 170\ 415$$

For the first two axes, it does not matter whether you choose the upper or the lower sign. For the third axes, however, in order to ensure that the principal axes form a right-handed set, choose the sign such that the determinant of the matrix of direction cosines is +1.

We have just seen that, if we know the moments and products of inertia *A, B, C, F, G, H* with respect to some axes (i.e. if we know the elements of the inertia tensor) we can find the principal moments of inertia A_0 , B_0 , C_0 by diagonalizing the inertia tensor, or finding its eigenvalues. If, on the other hand, we know the principal moments of inertia of a system of particles (or of a solid body, which is a collection of particles), how can we find the moment of inertia *I* about an axis whose direction cosines with respect to the principal axes are (l , m , n)?

First, some geometry.

Let O*xyz* be a coordinate system, and let P (x , y , z) be a point whose position vector is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Let L be a straight line passing through the origin, and let the direction cosines of this line be (l , m , n). A unit vector **e** directed along L is represented by

$$\mathbf{e} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}.$$

The angle θ between **r** and **e** is found from the scalar product $\mathbf{r} \bullet \mathbf{e}$, given by

$$r \cos\theta = \mathbf{r} \bullet \mathbf{e}.$$

I.e. $$\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\cos\theta = lx + my + nz.$$

The perpendicular distance p from P to L is

$$p = r\sin\theta = \left(x^2+y^2+z^2\right)^{\frac{1}{2}}\sin\theta.$$

If we write $\sin\theta = \left(1-\cos^2\theta\right)^{\frac{1}{2}}$, we soon obtain

$$p^2 = x^2 + y^2 + z^2 - (lx+my+nz)^2.$$

Noting that $l^2 = 1 - m^2 - n^2$, $m^2 = 1 - n^2 - l^2$, $n^2 = 1 - l^2 - m^2$, we find, after further manipulation:

$$p^2 = l^2\left(y^2+z^2\right)+m^2\left(z^2+x^2\right)+n^2\left(x^2+y^2\right)-2\left(mnyz+nlzx+lmxy\right).$$

Now return to our collection of particles, and let O*xyz* be the principal axes of the system. The moment of inertia of the system with respect to the line L is

$$I = \sum Mp^2,$$

where I have omitted a subscript i on each symbol. Making use of the expression for *p* and noting that the product moments of the system with respect to O*xyz* are all zero, we obtain

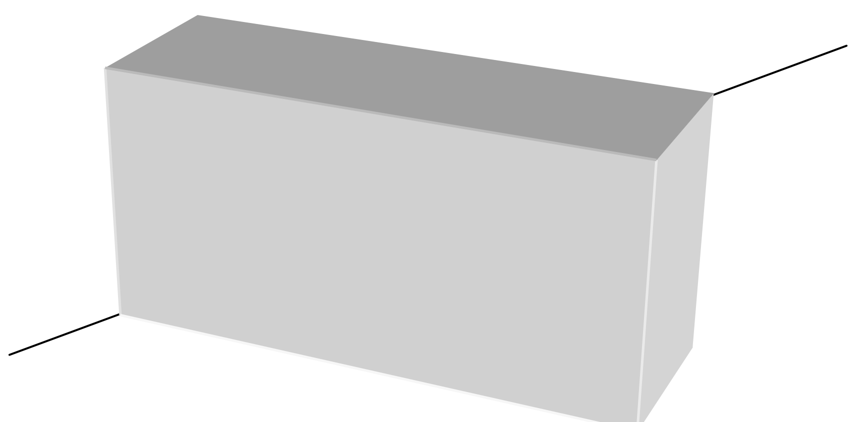
$$I = l^2 A_0 + m^2 B_0 + n^2 C_0. \qquad 2.18.1$$

Also, let *A, B, C, F, G, H* be the moments and products of inertia with respect to a set of nonprincipal orthogonal axes; then the moment of inertia about some other axis with direction cosines *l*, *m, n* with respect to these nonprincipal axes is

$$I = l^2A + m^2B + n^2C - 2mnF - 2nlG - 2lmH. \qquad 2.18.2$$

Example. A Brick.

We saw in section 16 that the moment of inertia of a uniform solid cube of mass *M* and side 2*a* about a body diagonal is $\frac{2}{3}Ma^2$, and we saw how very easy this was. At that time the problem of finding the moment of inertia of a uniform solid rectangular parallelepiped of sides 2*a*, 2*b*, 2*c* must have seemed intractable, but by now it is not at all hard.



$$A_0 = \tfrac{1}{3}M\left(b^2 + c^2\right)$$
$$B_0 = \tfrac{1}{3}M\left(c^2 + a^2\right)$$
$$C_0 = \tfrac{1}{3}M\left(a^2 + b^2\right)$$

Thus we have:

$$l = \frac{a}{\left(a^2 + b^2 + c^2\right)^{\frac{1}{2}}}$$
$$m = \frac{b}{\left(a^2 + b^2 + c^2\right)^{\frac{1}{2}}}$$
$$n = \frac{c}{\left(a^2 + b^2 + c^2\right)^{\frac{1}{2}}}.$$

We obtain:
$$I = \frac{2M\left(b^2c^2 + c^2a^2 + a^2b^2\right)}{3\left(a^2 + b^2 + c^2\right)}.$$

We note:

i. This is dimensionally correct;
ii. It is symmetric in a, b, c;
iii. If $a = b = c$, it reduces to $\frac{2}{3}Ma^2$.

2.19 *Moment of Inertia with Respect to a Point.*

By "moment of inertia" we have hitherto meant the second moment of mass *with respect to an axis.* We were easily able to identify it with the *rotational inertia* with respect to the axis, namely the ratio of an applied torque to the resulting angular acceleration.

I am now going to define the (second) moment of inertia with respect to a *point*, which I shall take unless otherwise specified to mean the origin of coordinates. If we have a collection of mass points m_i at distances r_i from the origin, I define

$$\mathcal{I} = \sum_i m_i r_i^2 = \sum_i m_i\left(x_i^2 + y_i^2 + z_i^2\right) \qquad 2.19.1$$

as the (second) moment of inertia with *respect to the origin*, also sometimes called the "geometric moment of inertia". I cannot relate it in an obvious way to a simple dynamical concept in the same way that I related moment of inertia with respect to an axis to rotational inertia, but we shall see that it is by no means merely a tedious exercise in arithmetic, and it does have its uses. The symbol *I* has probably been used rather a lot in this chapter; so to describe the geometric moment of inertia I am going to use the symbol $\mathcal{I}$.

The moment of inertia with respect to the origin is clearly something that does not depend on the orientation of any particular basis set of orthogonal axes, since it depends only on the distances of the particles from the origin.

If you recall the definitions of A, B and C from section 2.15, you will easily see that

$$\mathcal{I} = \tfrac{1}{2}(A + B + C). \tag{2.19.2}$$

and we already noted (see equation 2.16.2) that $A + B + C$ is invariant under rotation of axes. In section 2.18 we expressed it slightly more generally by saying "the trace of a symmetric matrix is invariant under an orthogonal transformation". By now it probably seems slightly less mysterious.

Let us now calculate the geometric moment of inertia of a uniform solid sphere of radius a, mass m, density ρ, with respect to the centre of the sphere. It is

$$\mathcal{I} = \int_{sphere} r^2 dm. \tag{2.19.3}$$

The element of mass, dm, here is the mass of a shell of radii r, $r + dr$; that is $4\pi\rho r^2 dr$. Thus

$$\mathcal{I} = 4\pi\rho \int_0^a r^4 dr = \tfrac{4}{5}\pi\rho a^5. \tag{2.19.4}$$

With $m = \frac{4}{3}\pi a^3 \rho$, this becomes

$$\mathcal{I} = \tfrac{3}{5} m a^2. \tag{2.19.5}$$

Indeed, for any *spherically symmetric* distribution of matter, since $A = B = C$, it will be clear from equation 2.19.2, that *the moment of inertia with respect to the centre is 3/2 times the moment of inertia with respect to an axis through the centre.* For example, it is obvious from the definition of moment of inertia with respect to the centre that for a hollow spherical shell it is just ma^2, and therefore the moment of inertia with respect to an axis through the centre is $\frac{2}{3}ma^2$. In other words, you can work out that the moment of inertia of a hollow spherical shell with respect to an axis through its centre is $\frac{2}{3}ma^2$ in your head without any of the integration that we did in section 2.7!

By way of illustration, consider three spheres, each of radius a and mass M, but the density between centre and surface varies as

$$\rho = \rho_0\left(1 - \frac{kr}{a}\right), \qquad \rho = \rho_0\left(1 - \frac{kr^2}{a^2}\right), \qquad \rho = \rho_0\sqrt{1 - \frac{kr^2}{a^2}}$$

for the three spheres. Calculate for each the moment of inertia about an axis through the centre of the sphere. Express the answer in the form $\frac{2}{5}Ma^2 \times f(k)$.

Solution. The mass of a sphere is

$$M = 4\pi\int_0^a \rho(r)r^2\,dr$$

and so

$$\tfrac{2}{5}Ma^2 = \frac{8\pi a^2}{5}\int_0^a \rho(r)r^2\,dr.$$

The moment of inertia about the centre is

$$\mathcal{J} = 4\pi\int_0^a \rho(r)r^4\,dr$$

and so the moment of inertia about an axis through the centre is

$$I = \frac{8\pi}{3}\int_0^a \rho(r)r^4\,dr.$$

Therefore

$$\frac{I}{\frac{2}{5}Ma^2} = \frac{5}{3a^2}\frac{\int_0^a \rho(r)r^4 dr}{\int_0^a \rho(r)r^2 dr}.$$

For the first two spheres the integrations are straightforward. I make it

$$\frac{I}{\frac{2}{5}Ma^2} = \frac{12-10k}{12-9k}$$

for the first sphere, and

$$\frac{I}{\frac{2}{5}Ma^2} = \frac{35-25k}{35-21k}$$

for the second sphere. The integrations for the third sphere need a little more patience, but I make the answer

$$\frac{I}{\frac{2}{5}Ma^2} = \frac{5(12\alpha - 3\sin 2\alpha - 3\sin 4\alpha + \sin 6\alpha)}{18\sin^2\alpha(4\alpha - \sin 4\alpha)},$$

where $\sin\alpha = \sqrt{k}$.

This should be enough to convince that the concept of $\mathcal{I}$ is useful – but it is not its only use. We shall meet it again in Chapter 3 on the dynamics of *systems of particles*; in particular, it will play a role in what we shall become familiar with as the *virial theorem.*

2.20 *Ellipses and Ellipsoids*

Here are some problems concerning ellipses and ellipsoids that might be of interest.

Determine the principal moments of inertia of the following:

1. A uniform plane lamina of mass m in the form of an ellipse of semi axes a and b.
2. A uniform plane ring of mass m in the form of an ellipse of semi axes a and b.
3. A uniform solid triaxial ellipsoid of mass m and semi axes a, b and c.
4. A uniform hollow triaxial ellipsoid of mass m and semi axes a, b and c.

1. By integration, an elliptical lamina is slightly difficult, but by physical insight it is very easy!

The distribution of mass around the minor axis is the same as for a circular lamina of radius a, and therefore the moment B is the same as for the circular lamina, namely $B = \frac{1}{4}ma^2$. Similarly, $A = \frac{1}{4}mb^2$, and hence, by the perpendicular axes theorem, $C = \frac{1}{4}m(a^2 + b^2)$.

I think you will find that the shape of the momental ellipse is the same as the shape of the original elliptical lamina.

2. An elliptical ring (hoop) is remarkably difficult. It cannot be expressed in terms of elementary functions, and it has to be calculated numerically. It can be expressed in terms of elliptic integrals (no surprise there), but most of us aren't sure what elliptic integrals are and they hardly count as elementary functions, and they have to be calculated numerically anyway. We take the ellipse to be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with $b \le a$.

Even calculating the circumference of an ellipse isn't all that easy. The circumference is

$$\oint ds = 4\int_0^a \left[1 + \left(\tfrac{dy}{dx}\right)^2\right]^{1/2} dx\,, \text{ with } y = b\left(1 - \frac{x^2}{a^2}\right)^{1/2}.$$

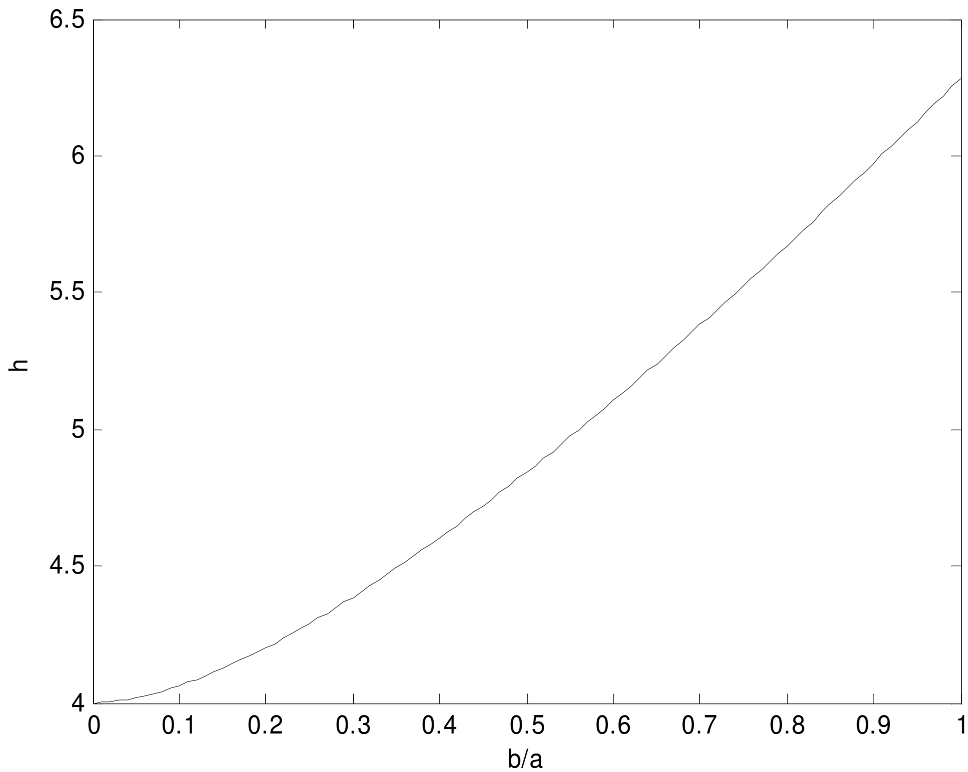
After a bit of algebra, this can be written as

$$\frac{4a}{c}\int_0^a \sqrt{\frac{c^2 - x^2}{a^2 - x^2}}\, dx\,, \quad \text{where } c^2 = \frac{a^4}{a^2 - b^2}.$$

At first this looks easy, but I don't think you can do it in terms of elementary functions. No problem, then – just integrate it numerically. Unfortunately the integrand becomes infinite at the upper limit, so there is still a bit of a problem. However, a change of variable to $x = a\sin\theta$ solves that problem. The expression for the circumference becomes simply

$$4a\int_0^{\pi/2}\left[1-\left(\frac{a^2-b^2}{a^2}\right)\sin^2\theta\right]^{1/2}d\theta,$$

which can be integrated numerically without infinity problems at the limits. According to my calculations, the circumference of the ellipse is ha, where h is a function of b/a as follows:



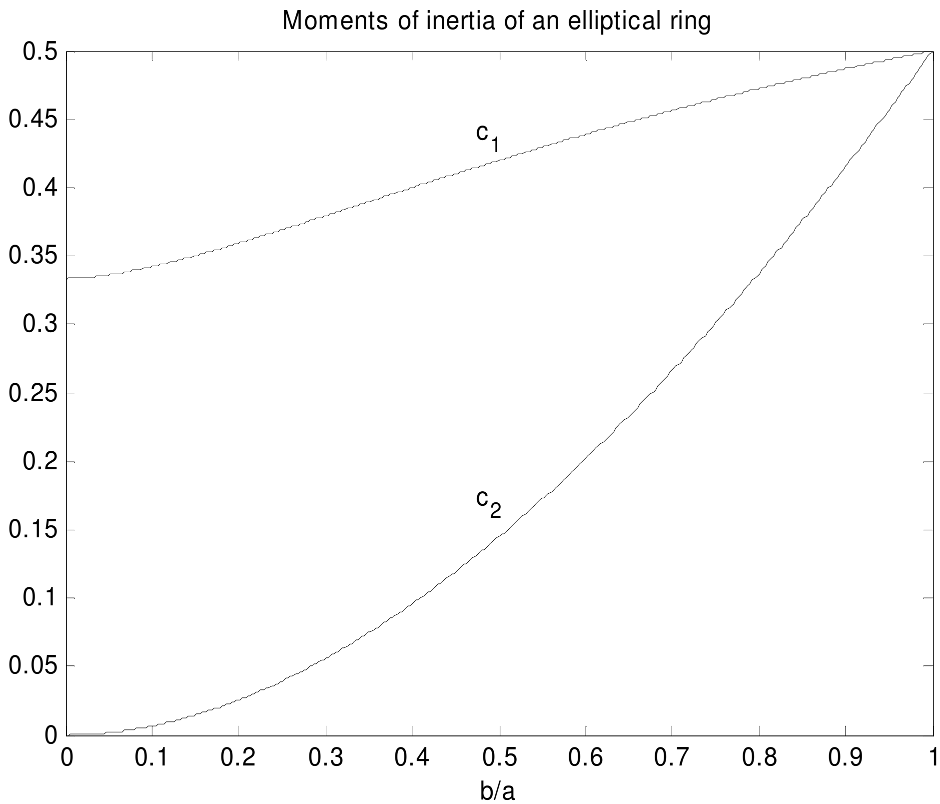
To find the moment of inertia (or the second moment of length) about the minor axis, we have to multiply the integrand by x^2, or $a^2\sin^2\theta$, and integrate. Thus the moment of inertia of the elliptical hoop about its minor axis is $c_1 ma^2$, where

$$c_1 = \frac{\displaystyle\int_0^{\pi/2}\left[1 - \left(\frac{a^2 - b^2}{a^2}\right)\sin^2\theta\right]^{1/2}\sin^2\theta d\theta}{\displaystyle\int_0^{\pi/2}\left[1 - \left(\frac{a^2 - b^2}{a^2}\right)\sin^2\theta\right]^{1/2}d\theta}.$$

The moment of inertia about the major axis is c_2ma^2, where

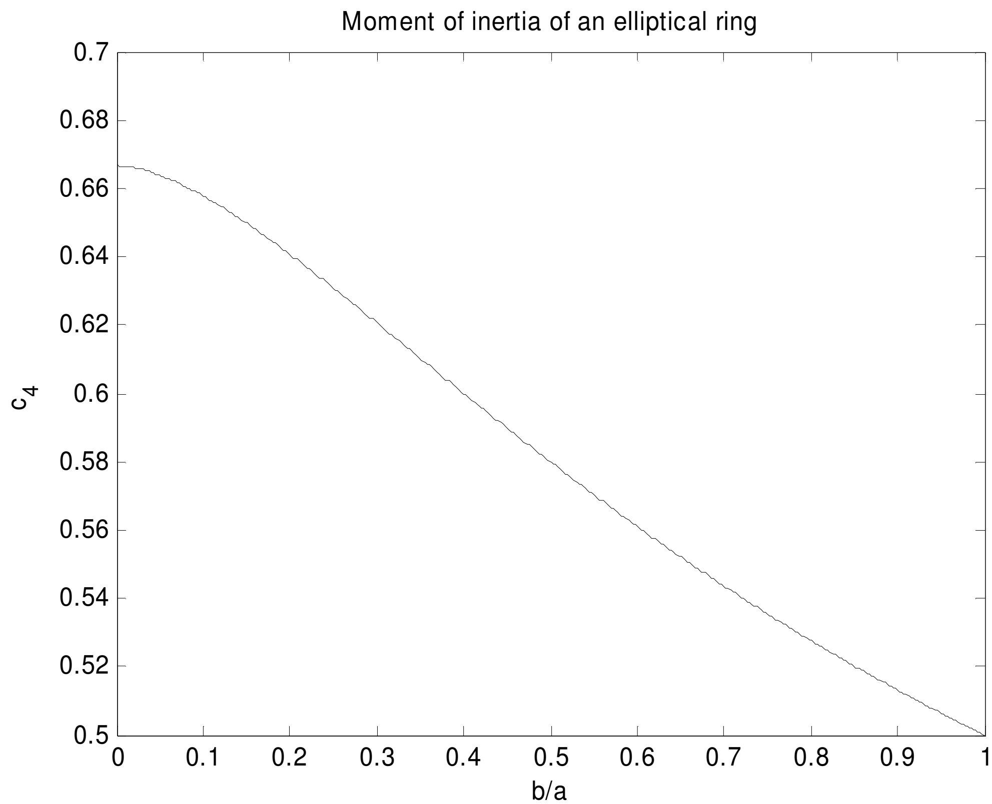
$$c_2 = \frac{\displaystyle\frac{b^2}{a^2}\int_0^{\pi/2}\left[1 - \left(\frac{a^2 - b^2}{a^2}\right)\cos^2\theta\right]^{1/2}\sin^2\theta d\theta}{\displaystyle\int_0^{\pi/2}\left[1 - \left(\frac{a^2 - b^2}{a^2}\right)\sin^2\theta\right]^{1/2}d\theta}.$$

These two coefficients of ma^2 are shown below as a function of *b*/*a*.



The moments of inertia of an elliptical ring of mass *m* and semi major and semi minor axes *a* and *b* are c_1ma^2 about the minor axis and c_2ma^2 about the major axis, where c_1 and c_2 are shown as functions of *b*/*a*.

The moment of inertia about the major axis can also be conveniently expressed in terms of b rather than a. If we write the moment of inertia about the major axis as $c_4 mb^2$, then c_4 as a function of b/a is shown below.



The moment of inertia of an elliptical ring of mass m and semi major and semi minor axes a and b is $c_4 mb^2$ about the major axis, where c_4 is shown as a function of b/a.

The moment of inertia about an axis perpendicular to the plane of the ellipse and passing through its centre is $c_3 ma^2$, where, of course (by the perpendicular axes theorem), $c_3 = c_1 + c_2$. It is also equal to $c_1 ma^2 + c_4 mb^2$.

3. For a uniform solid triaxial ellipsoid, the moments of inertia are

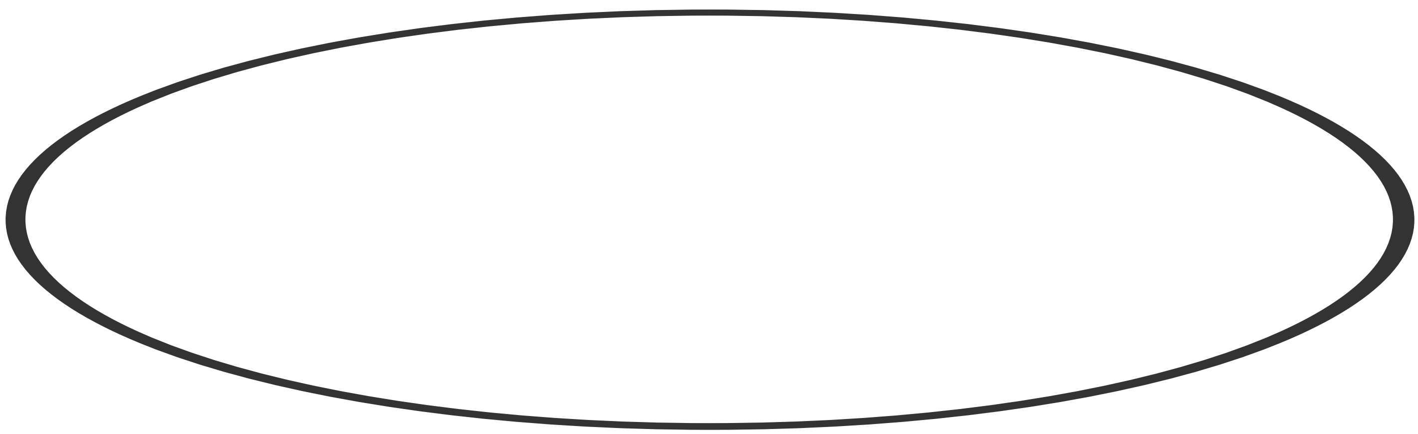
$$A = \tfrac{1}{5}m(b^2 + c^2) \qquad B = \tfrac{1}{5}m(c^2 + a^2) \qquad C = \tfrac{1}{5}m(a^2 + b^2)$$

The momental ellipsoid is not of the same shape. Its axes are in the ratio

$$1 : \sqrt{\frac{b^2 + c^2}{c^2 + a^2}} : \sqrt{\frac{b^2 + c^2}{a^2 + b^2}} \; .$$

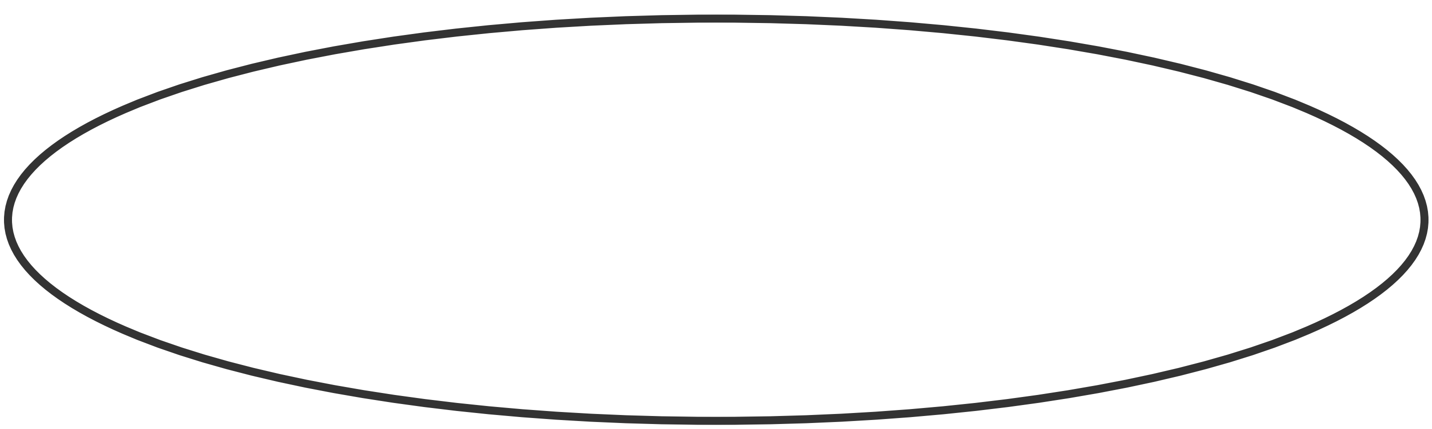
For example, if the axial ratios of the original ellipsoid are 1 : 2 : 3, the axial ratios of the corresponding momental ellipsoid is $1 : \sqrt{\frac{13}{10}} : \sqrt{\frac{13}{5}} = 1 : 1.140 : 1.612$, which is slightly more spherical than the original ellipsoid.

4. Triaxial elliptical shell. We have to think carefully about what a triaxial elliptical shell is. If we imagine the inner surface of the shell to be an ellipsoid, and the outer surface to be a similar ellipsoid, but with all linear dimensions increased by the same small fractional increment, then we obtain a figure like this:



In this drawing the linear size of the outer surface is 3 percent larger than that of the inner surface. E. J. Routh correctly shows in his treatise on rigid bodies that the principal moments of inertia of such a figure are $\frac{1}{3}m(b^2 + c^2)$, $\frac{1}{3}m(c^2 + a^2)$, $\frac{1}{3}m(a^2 + b^2)$.

But it can be seen that such a figure is not (as presumably a rugger ball is) of *uniform thickness*. I draw below a shell of uniform thickness. In such a case the inner and outer surfaces are not exactly similar.



In attempting to calculate the moment of inertia of such a figure I shall restrict myself to the case of a *spheroidal* shell of uniform thickness. That is to say, an ellipsoid with two equal axes, represented by the equation, in cylindrical coordinates

$$\frac{\rho^2}{a^2} + \frac{z^2}{c^2} = 1,$$

where $\rho^2 = x^2 + y^2$. Further, if I put $c = \chi a$, the equation to the spheroid can be written

$$\rho^2 + \frac{z^2}{\chi^2} = a^2.$$

If $\chi < 1$, the spheroid is *oblate.* If $\chi > 1$, the spheroid is *prolate.*

We'll first need to calculate its surface area, which is

$$A = 4\pi \int_0^c \rho \left[1 + \left(\frac{d\rho}{dz}\right)^2\right]^{1/2} dz.$$

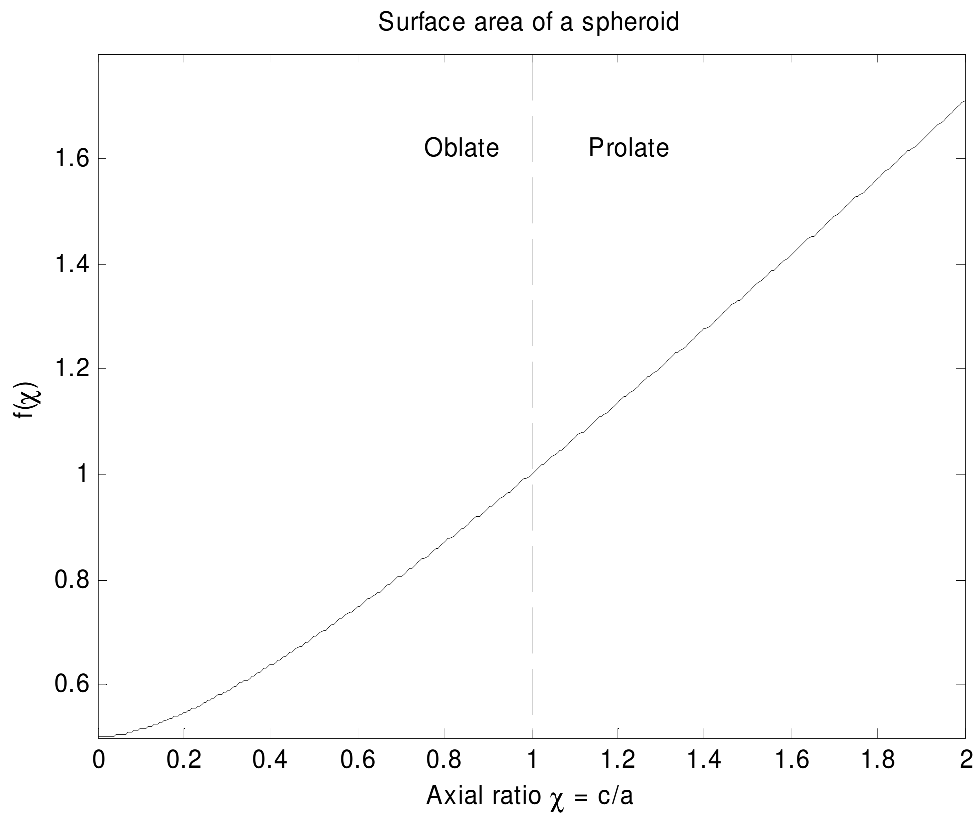
After some algebra, this comes to

$$A = 4\pi a^2 f(\chi),$$

where $$f(\chi) = \tfrac{1}{2}\left[\frac{\chi^2}{\sqrt{1-\chi^2}} \ln\left(\frac{1+\sqrt{1-\chi^2}}{\chi}\right) + 1\right] \quad \text{for } \chi \le 1$$

and $$f(\chi) = \tfrac{1}{2}\left[\frac{\chi^2}{\sqrt{\chi^2-1}} \sin^{-1}\left(\frac{\sqrt{\chi^2-1}}{\chi}\right) + 1\right] \quad \text{for } \chi \ge 1.$$

This function is shown below as far as $\chi = 2$. For $\chi = 0$, the figure is a disc whose total area (upper and lower surface) is $2\pi a^2$, and $f = \tfrac{1}{2}$. For $\chi = 1$, the figure is a sphere whose area is $4\pi a^2$, and $f = 1$. The function goes to infinity as χ goes to infinity.



The moment of inertia about the *z*-axis is

$$I = \frac{4\pi m}{A}\int_0^c \rho^3\left[1 + \left(\frac{d\rho}{dz}\right)^2\right]^{1/2} dz.$$

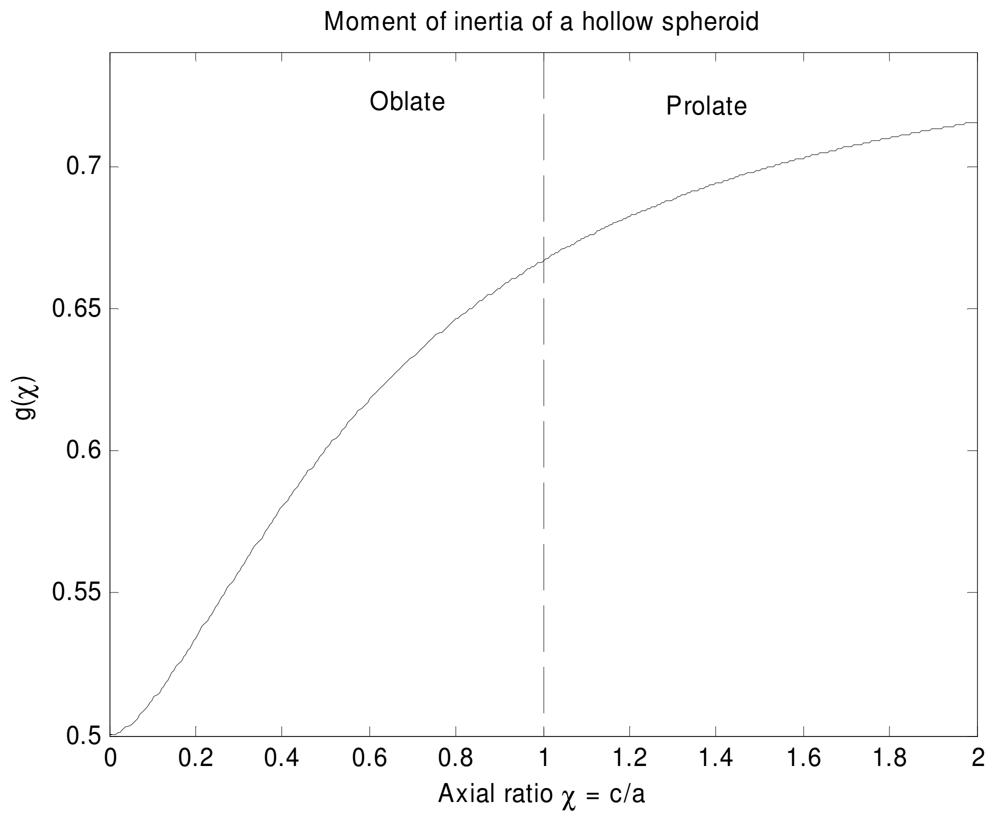
After some algebra this becomes

$$I = ma^2 g(\chi),$$

$$g(\chi) = 1 - \frac{(2-\chi^2)(1-\chi^2) - \chi^4 \ln\left[\left(1+\sqrt{1-\chi^2}\right)/\chi\right]}{4\left\{(1-\chi^2)^{3/2} + \chi^2(1-\chi^2)\ln\left[\left(1+\sqrt{1-\chi^2}\right)/\chi\right]\right\}} \quad \text{for} \quad \chi \le 1$$

$$g(\chi) = 1 - \frac{\dfrac{\chi^4}{(\chi^2-1)^{3/2}}\sin^{-1}\left(\dfrac{\sqrt{\chi^2-1}}{\chi}\right) + \dfrac{\chi^2-2}{\chi^2-1}}{4\left\{\dfrac{\chi^2}{\sqrt{\chi^2-1}}\sin^{-1}\left(\dfrac{\sqrt{\chi^2-1}}{\chi}\right) + 1\right\}} \quad \text{for} \quad \chi \geq 1.$$

This function is shown below as far as $\chi = 2$. For $\chi = 0$, the figure is a disc whose moment of inertia is $\frac{1}{2}\pi a^2$, and $f = \frac{1}{2}$. For $\chi = 1$, the figure is a hollow sphere whose moment of inertia is $\frac{2}{3}\pi a^2$, and $f = \frac{2}{3}$. The function goes to 1 as χ goes to infinity; the moment of inertia then approaches that of a hollow cylinder.



2.21 *Tetrahedra*

Exercise. Show that the moment of inertia about an axis through the centre of mass of a uniform solid regular tetrahedron of mass m and edge length a is $\frac{1}{20}ma^2$.

Exercise. Show that the moment of inertia of a methane molecule about an axis through the carbon atom is $\frac{8}{3}ml^2$, where l is the bond length and m is the mass of a hydrogen atom.

And, in case you are wondering that I haven't specified the *orientation* of the axis in either case, the solid regular tetrahedron and the methane molecule are both spherical tops, and the moment of inertia is the same about *any* axis through the centre of mass.