CHAPTER 5
ABSORPTION, SCATTERING, EXTINCTION AND THE EQUATION OF TRANSFER

5.1 *Introduction*

As radiation struggles to make its way upwards through a stellar atmosphere, it may be weakened by *absorption* and *scattering*. The combined effect of absorption and scattering is called *extinction*. Scattering may simply be by reflection from dust particles. If the radiation interacts with an *atom*, the atom may be excited to a higher energy level and almost immediately (typically on a time-scale of nanoseconds) the atom drops down to its original level and emits a photon of the same frequency as the one it absorbed. Such a process - temporary absorption followed almost immediately by re-emission without change in wavelength - is probably best described in the present context as scattering. Individual atoms in a stellar atmosphere generally radiate dipole radiation; however, since many randomly oriented atoms take place in the process, the scattering can be regarded as isotropic. If, however, the excited atom collides with another atom before re-emission, the collision may be super-elastic; as the atom falls to a lower state, the energy it gives up, instead of being radiated as a photon, goes to kinetic energy of the colliding atoms. The radiation has been converted to kinetic energy. This process is absorption.

5.2 *Absorption*

To start with, let us suppose that the predominating mechanism is absorption with no scattering. We can define a *linear absorption coefficient* α as follows. Let the specific intensity at some level in an atmosphere be I. At a level in the atmosphere higher by a distance dx, the specific intensity has dropped, as a result of absorption, to $I + dI$. (Here dI, by the convention of differential calculus, means the <u>increase</u> in I, and it is in this case negative. The quantity $-dx$, which is positive, is the <u>decrease</u> in I.) The linear absorption coefficient α is defined such that the fractional decrease in the specific intensity over a distance dx is given by

$$-\frac{dI}{I} = \alpha dx \tag{5.2.1}$$

The coefficient is of dimension L^{-1} and the SI unit is m^{-1}. In general, α will depend on frequency or wavelength, and, at a particular wavelength, the equation would be written

$$-\frac{dI_\nu}{I_\nu} = \alpha(\nu)dx \tag{5.2.2}$$

If equation 5.2.1 is integrated over a finite distance, for a slab of atmosphere, say, between $x = 0$, where the specific intensity is I^0, and $x = X$, where the specific intensity is I, it becomes

$$I = I^0 \exp\left[-\int_0^X \alpha(x)dx\right] \qquad 5.2.3$$

And if α is uniform and not a function of *x*, this becomes

$$I = I^0 \exp(-\alpha X) \qquad 5.2.4$$

Now let $\alpha_a = \alpha/n$, so that equation 5.2.1 becomes $-dI/I = \alpha_a n dx$ and equation 5.2.4 becomes $I = I^0 \exp(-\alpha_a nX)$, where n is the number of atoms per unit volume. Then α_a is the *atomic absorption coefficient,* or *atomic absorption cross-section.* It is of dimension L^2 and the SI unit is m^2.

In a similar manner, we can define $\alpha_m = \alpha/\rho$, where ρ is the mass density, as the *mass absorption coefficient,* with corresponding modifications in all the other equations. It is of dimension L^2M^{-1} and the SI unit is $m^2\ kg^{-1}$.

We might also mention here that in laboratory chemistry, one comes across the word *absorbance* of a solution. This is the linear absorption coefficient divided by the concentration of the solute. While this word in not usually encountered in stellar atmosphere theory, it is mentioned here partly because it is very similar in concept to the several concepts discussed in this section, and also because of the similarity of the word to the rather different *absorptance* defined in Chapter 2. In chemical texts, the exponential decrease of intensity with distance is often referred to as the Lambert-Beer Law, or simply as Lambert's Law. This is mentioned here merely to point out that this is not at all related to the Lambert's Law discussed in Chapter 1.

5.3 *Scattering, Extinction and Opacity*

If the predominating mechanism is scattering with no absorption, we can define in a similar manner linear, atomic and mass scattering coefficients, using the symbol σ rather than α. For the physical distinction between absorption and scattering, see section 5.1. And if both absorption and scattering are important, we can define linear, atomic and mass extinction coefficients, using the symbol κ, where $\kappa = \alpha + \sigma$.

All the foregoing equations are valid, whether we use linear, atomic or mass absorption, scattering or extinction coefficients, and whether we refer to radiation integrated over all frequencies or whether at a particular wavelength or within a specified wavelength range.

The *mass extinction coefficient* is generally referred to as the *opacity.*

5.4 *Optical depth*

The product of linear extinction coefficient and distance, or, more properly, if the extinction coefficient varies with distance, the integral of the extinction coefficient with respect to distance, $\int \kappa(x)\,dx$, is the *optical depth,* or *optical thickness,* τ. It is dimensionless. Specific intensity falls off with optical depth as $I = I^0 e^{-\tau}$. Thus optical depth can also be defined by ln (I^0/I). While the optical depth ln (I^0/I) is generally used to describe how opaque a stellar atmosphere or an interstellar cloud is, when describing how opaque a filter is, one generally uses $\log_{10}$ (I^0/I), which is called the *density d* of the filter. Density is 0.4343 times optical depth. If a star is hidden behind a cloud of optical depth τ it will be dimmed by 1.086τ magnitudes. If it is hidden behind a filter of density d it will be dimmed by $2.5d$ magnitudes. The reader is encouraged to verify these assertions.

5.5 *The Equation of Transfer.*

The equation of transfer deals with the transfer of radiation through an atmosphere that is simultaneously absorbing, scattering and emitting.

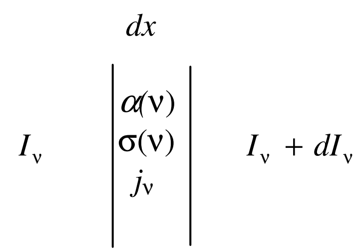

FIGURE V.1

Suppose that, between x and $x + dx$ the absorption coefficient and the scattering coefficient at frequency ν are $\alpha(\nu)$ and $\sigma(\nu)$, and the emission coefficient per unit frequency interval is $j_\nu d\nu$. In this interval, suppose that the specific intensity per unit frequency interval increases from I_ν to $I_\nu + dI_\nu$ (dI_ν might be positive or negative). The specific intensity will be reduced by absorption and scattering and increased by emission. Thus:

$$dI_\nu = -\left[I_\nu\alpha(\nu) + I_\nu\sigma(\nu) - j_\nu(\nu)\right]dx. \qquad 5.5.1$$

This is one form - the most basic form - of the *equation of transfer.* Notice that α and σ do not have a subscript.

5.6 *The Source Function* (**Die Ergiebigkeit**)

This is the ratio of the emission coefficient to the extinction coefficient. A review of the dimensions of these will show that the dimensions of source function are the same as that of specific intensity, namely W m^{-2} sr^{-1} (perhaps per unit wavelength or frequency interval). The usual symbol is *S*. Thus

$$S_\nu = \frac{j_\nu}{\alpha(\nu) + \sigma(\nu)} = \frac{j_\nu}{\kappa(\nu)} \qquad 5.6.1$$

Imagine a slice of gas of thickness *dx*. Multiply the numerator and denominator of the right hind side of equation 5.6.1 by *dx*. Observe that the numerator is now the specific intensity (radiance) of the slice, while the denominator is its optical thickness. Thus an alternative definition of source function is *specific intensity per unit optical thickness.* Later, we shall evaluate the source function in an atmosphere in which the extinction is pure absorption, in which it is purely scattering, and in which it is a bit of each.

5.7 *A Series of Problems*

I am now going to embark upon a series of problems that at first sight may appear to be not very relevant to stellar atmospheres, but the reader is urged to be patient and look at them, partly because they make use of many of the ideas encountered up to this point, and also because they culminate in determining how the flux and the mean specific intensity in an atmosphere increase with optical depth in terms of the source function.

Problem 1

An infinite plane radiating surface has a uniform specific intensity (radiance) *I.* What is the flux (irradiance) at a point P, situated at a height *h* above the surface?

We have already answered that question in equation1.15.3, and the answer, which, unsurprisingly since the plane is infinite in extent, is independent of *h*, is πI, so let's get on with

Problem 2

Same as Problem 1, except that this time the space between the radiating plane and the point P is filled with a uniform gas of absorption coefficient α. The specific intensity (radiance) of the surface, we are told, is, following astrophysical custom, *I.* Unfortunately I shall also be compelled to make use of "intensity" in the "standard" sense of Chapter 1, and for that I shall use the symbol $\mathfrak{I}$.

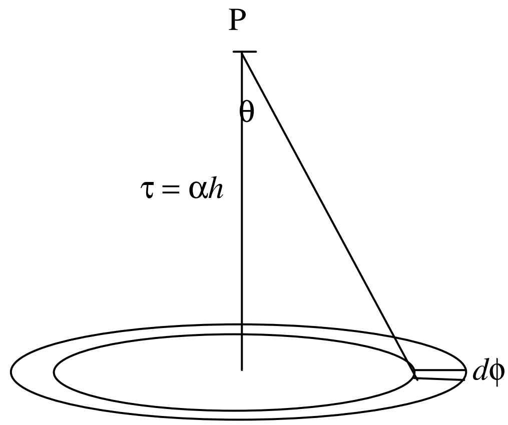

FIGURE V.2

The elemental area is $r\,dr\,d\phi$, or, since $r = h \tan\theta$, it is $h^2 \tan\theta \sec^2\theta\, d\theta d\phi$.
The intensity of the elemental area towards P is the specific intensity (radiance) times the projected area:

$$d\mathcal{I} = Ih^2 \tan\theta \sec^2\theta\, d\theta\, d\phi \cos\theta$$

If there were no absorption, the irradiance of P by the elemental area would be

$$d\mathcal{I} \cos\theta \;/\; (h^2 \sec^2\theta),$$

which becomes $I \sin\theta \cos\theta\, d\theta\, d\phi$.

But it is reduced by absorption by a factor $e^{-\tau\sec\theta}$, where $\tau = \alpha h$. Therefore the irradiance of P by the elemental area is

$$Ie^{-\tau\sec\theta} \sin\theta \cos\theta\, d\theta\, d\phi.$$

For the irradiance at P (or "flux" in the astrophysics sense) by the entire infinite plane we integrate from $\phi = 0$ to 2π and $\theta = 0$ to $\pi/2$, to obtain

$$2\pi I \int_0^{\pi/2} e^{-\tau\sec\theta} \sin\theta \cos\theta\, d\theta$$

If we now write $x = \sec\theta$, this becomes

$$\text{Irradiance at P} = 2\pi I E_3(\tau),$$

and we hope that the reader has not forgotten the meaning of E_3 - if you have, as the game of snakes and ladders would say, Go back to Chapter 3. Note that, at $\tau = 0$, this becomes πI, as expected.

Problem 3

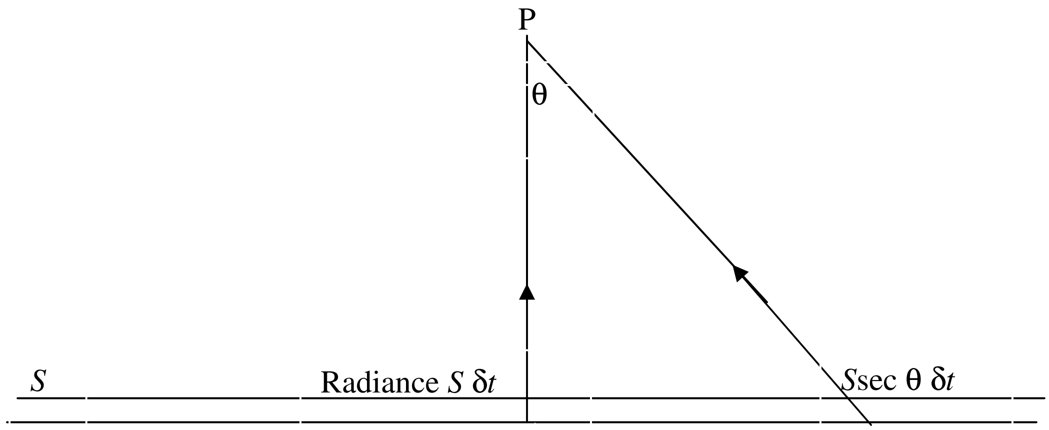

FIGURE V.3

A point P is situated at a height h above an infinite plane slice of gas of optical thickness $\delta\tau$ and source function S. There is nothing between P and the slice of gas. What is the flux (irradiance) at P?

At first glance this appears to be identical to Problem 1, except that the specific intensity of the slice is $S\delta t$. However, a more careful look at the diagram will reveal that the specific intensity of the slice is by no means uniform. It is darkest directly below P, and, when P looks farther from his nadir, the slice gets brighter and brighter, being $S \sec\theta\, \delta t$ at an angle θ. The upwards flux ("irradiance") at P is therefore

$$F_+ = 2\pi S\delta t \int_0^{\pi/2} \sec\theta \cos\theta \sin\theta \, d\theta = 2\pi S\delta t$$

Problem 4

Same as Problem 3, except that this time we'll place an absorbing gas of optical thickness t between P and the slice δt.

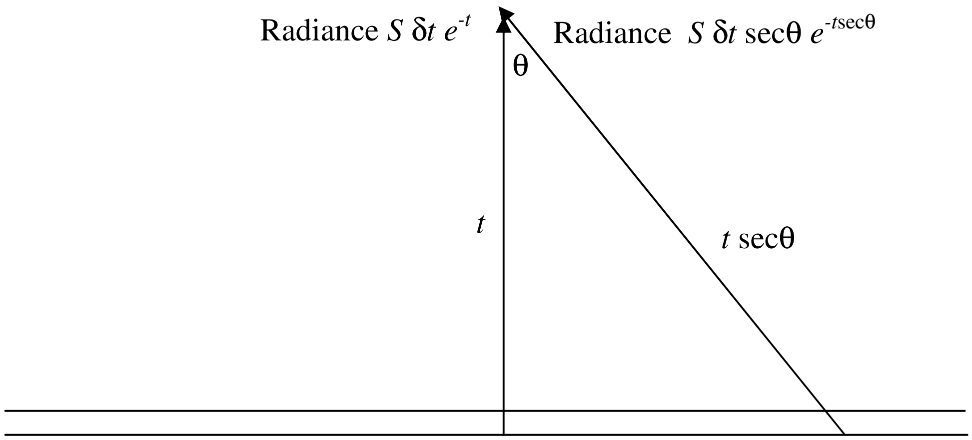

FIGURE V.4

In that case the flux (irradiance) at P from an element at an angle θ is reduced by $e^{-t\sec\theta}$ and consequently the flux at P from the entire slice is

$$F_+ \;=\; 2\pi S\delta t\int_0^{\pi/2} \sec\theta\cos\theta\sin\theta\, d\theta = 2\pi S\delta t.$$

If we write $x = \sec\theta$, we very soon see that this is

$$\text{Flux (irradiance) at P} = 2\pi S\delta t E_2(t)$$

Problem 5 (an important result in atmosphere theory)

Now consider a point P at an optical depth τ in a stellar atmosphere. (The use of the word "depth" will imply that τ is measured downwards from the surface towards the centre of the star.) We shall assume a plane parallel atmosphere i.e. a shallow atmosphere, or one than is shallow compared with the radius of the star, or we are not going to go very deep into the atmosphere. The point P is embedded in an absorbing, scattering, emitting gas. The flux coming up from below is equal to contributions from all the slices beneath P, from $t = \tau$ to $t = \infty$:

$$F_+ \;=\; 2\pi\int_\tau^\infty S(t)E_2(t-\tau)dt$$

The flux pouring down from above is the contribution from all the slices above, from $t = 0$ to $t = \tau$:

$$F_{-} = 2\pi \int_0^{\tau} S(t) E_2(\tau - t)\, dt$$

The net upward flux at a point P at an optical depth τ in an absorbing, scattering, emitting atmosphere is

$$F(\tau) = 2\pi \left[\int_{\tau}^{\infty} S(t) E_2(t-\tau)\, dt - \int_0^{\tau} S(t) E_2(\tau - t)\, dt \right] \qquad 5.7.1$$

The integral H is just $1/(4\pi)$ times this.

The reader is now asked to find the integrals $J(\tau)$ and $K(\tau)$. These should be given in the form of integrals that include a source function $S(t)$ and an exponential integral function $E(t - \tau)$ or $E(\tau - t)$. It is important to get the argument the right way round. One way is right; the other is wrong.

Problem 6

This is an easier problem, though the result is nevertheless important.

FIGURE V.5

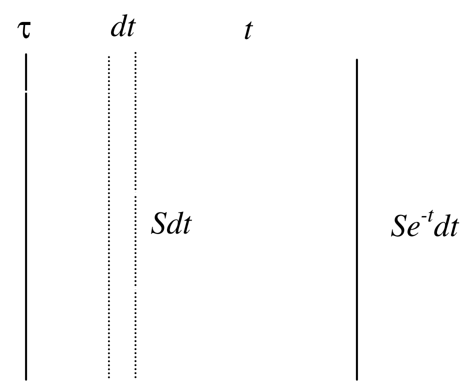

Figure V.5 shows a slab of gas of optical thickness τ. The observer is supposed to be to the right of the slab, and optical depth is measured from the right hand face of the slab towards the left. At an optical depth t within the slab is a slice of optical thickness dt. The slab is supposed to have a uniform source function S throughout. Source function is specific intensity per unit optical thickness, so the specific intensity of the slice is Sdt. The emergent intensity from this slice, by the time that it reaches the right hand surface of the slab, is $Se^{-t}dt$. The emergent specific intensity of the entire slab is the sum of the contributions of all such slices throughout the slab; that is $\int_0^{\tau} Se^{-t}\, dt$. If the source function is uniform throughout the slab, so that S is not a function of t, we find that the emergent specific intensity of the slab is

$$I = S\left(1 - e^{-\tau}\right) \qquad 5.7.2$$

Problem 7. A quantity of hot gas is held in a box 50 cm long. The emission coefficient of the gas is 0.06 W sr^{-1} m^{-3} and the extinction coefficient is 0.025 cm^{-1}. What is the emergent specific intensity (radiance)? (I make it 1.71×10^{-2} W m^{-2} sr^{-1}.)

5.8 *Source function in scattering and absorbing atmospheres.*

Suppose that at some point in a stellar atmosphere the mean specific intensity per unit frequency interval surrounding it is J_ν. If all of the radiation arriving at that point is isotropically scattered, the emission coefficient j_ν will simply be $\sigma(\nu)J_\nu$. But from equation 5.6.1 we see that in a purely scattering atmosphere, the ratio of j_ν to $\sigma(\nu)$ is the source function. Thus we see that, for an atmosphere in which the extinction is due solely to scattering, the source function is just

$$S_\nu = J_\nu . \qquad 5.8.1$$

If on the other hand the extinction is all due to absorption, we have $S_\nu = j_\nu/\alpha(\nu)$. If we multiply top and bottom by *dx,* the numerator will be dI_ν, the increase in the specific intensity in a distance dx, while the denominator is the absorptance in a layer of thickness *dx.* Thus the source function in a purely absorbing atmosphere is the ratio of the specific intensity to the absorptance. But this ratio is the same for all surfaces, including that of a black body, for which the absorptance is unity. Thus in an atmosphere in which the extinction is due solely to absorption, the source function is equal to the specific intensity (radiance) of a black body, for which we shall use the symbol *B*. For a purely absorbing atmosphere, we have

$$S_\nu = B_\nu . \qquad 5.8.2$$

In an atmosphere in which extinction is by both scattering and absorption the source function is a linear combination of equations 5.8.1 and 5.8.2, in proportion to the relative importance of the two processes:

$$S_\nu = \frac{\alpha(\nu)}{\alpha(\nu)+\sigma(\nu)} B_\nu + \frac{\sigma(\nu)}{\alpha(\nu)+\sigma(\nu)} J_\nu \qquad 5.8.3$$

5.9 *More on the equation of transfer.*

Refer to equation 5.5.1. We see from what had been subsequently discussed that $[\alpha(\nu) + \sigma(\nu)]dx = d\tau(\nu)$ and that $j_\nu \, dx = d\tau(\nu)$. Therefore

$$\frac{dI_\nu}{d\tau(\nu)} = S_\nu - I_\nu , \qquad 5.9.1$$

and this is another form of the *equation of transfer.*

Now consider a spherical star with a shallow atmosphere ("plane parallel atmosphere"). In figure V.6, radial distance *r* is measured radially outwards from the centre of the star. Optical depth is measured from outside towards the centre of the star. The thickness of the layer is *dr*. The coordinate *z* is measured from the centre of the star towards the observer, and the path length through the atmosphere in that direction at angle θ is $dz = dr\sec\theta$. The equation of transfer can be written

$$dI_\nu(\theta) = -\left[\kappa(\nu)I_\nu(\theta) - j_\nu\right]dz. \qquad 5.9.2$$

Now $\kappa(\nu)dz = -\sec\theta d\tau(\nu)$ and $j_\nu = \kappa(\nu)S_\nu$. Therefore

$$\cos\theta\frac{dI_\nu(\theta)}{d\tau(\nu)} = I_\nu(\theta) - S_\nu \qquad 5.9.3$$

This is yet another form of the equation of transfer. The quantity cosθ is often written μ, so that equation 5.9.3 is often written

$$\mu\frac{dI_\nu(\theta)}{d\tau(\nu)} = I_\nu(\theta) - S_\nu \qquad 5.9.4$$

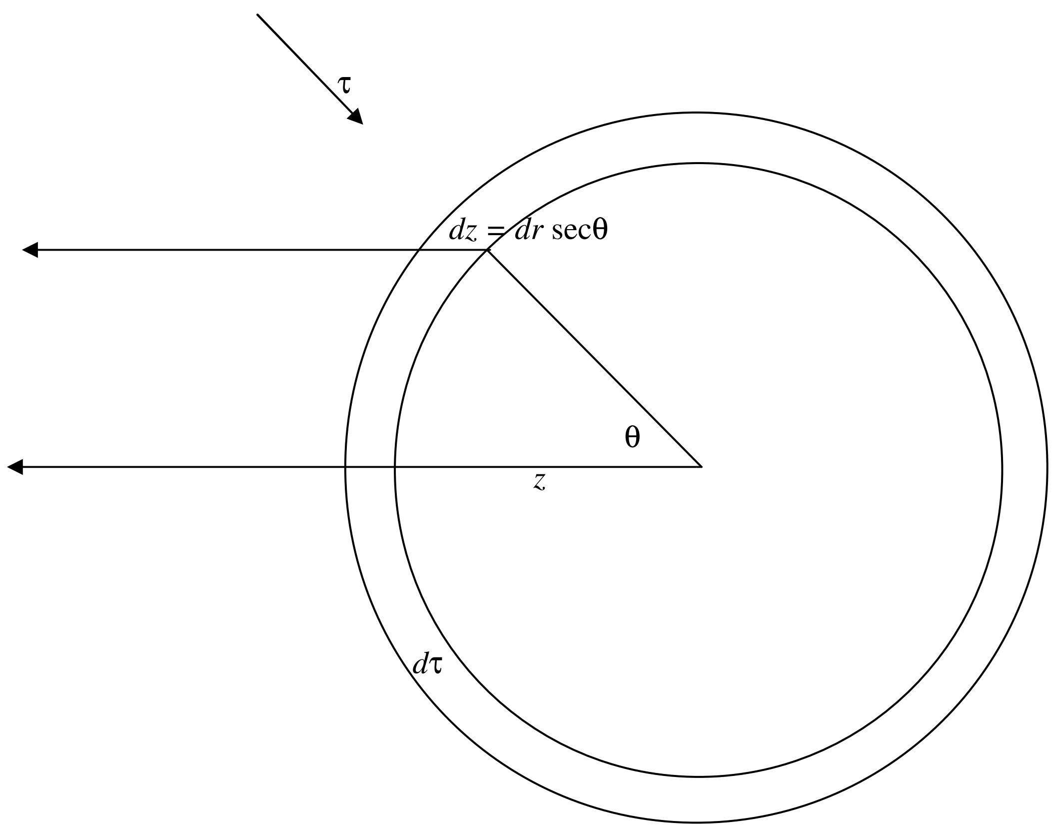

FIGURE V.6

Let us do $\frac{1}{4\pi}\oint d\omega$ to each term in equation 5.9.4. By $\oint$ I mean integrate over 4π steradians. In spherical coordinates $d\omega = \sin\theta d\theta d\phi$. We obtain

$$\frac{1}{4\pi}\oint \frac{dI_\nu(\theta)}{d\tau(\nu)}\cos\theta d\omega = \frac{1}{4\pi}\oint I_\nu d\omega - \frac{1}{4\pi}\oint S_\nu d\omega \qquad 5.9.5$$

The left hand side is $dH_\nu/d\tau(\nu)$ and the first term on the right hand side is J_ν. (See the definitions - equations 4.5.2 and 4.7.1.) In the case of isotropic scattering, the source function is isotropic so that, in this case

$$\frac{dH_\nu}{d\tau(\nu)} = J_\nu - S_\nu, \qquad 5.9.6$$

and this is another form of the equation of transfer.

On the other hand, if we do $\frac{1}{4\pi}\oint \cos\theta d\omega$ to each term in equation 5.15, we obtain

$$\frac{1}{4\pi}\oint \frac{dI_\nu(\theta)}{d\tau(\nu)}\cos^2\theta d\omega = \frac{1}{4\pi}\oint I_\nu \cos\theta d\omega - \frac{1}{4\pi}\oint S_\nu \cos\theta d\omega \qquad 5.9.7$$

In the case of isotropic scattering the last integral is zero, so that

$$\frac{dK_\nu}{d\tau(\nu)} = H_\nu, \qquad 5.9.8$$

and this is yet another form of the equation of transfer.

Now H_ν is independent of optical depth (why? - in a plane parallel atmosphere, this just expresses the fact that the flux (watts per square metre) is conserved), so we can integrate equation 5.9.8 to obtain

$$K_\nu = H_\nu\tau(\nu) + \text{constant}. \qquad 5.9.9$$

Note also that $H_\nu = F_\nu/(4\pi)$, and, if the radiation is isotropic, $K_\nu = J_\nu/3$, so that

$$J_\nu = \frac{3F_\nu\tau(\nu)}{4\pi} + J_\nu(0) \qquad 5.9.10$$

where $J_\nu(0)$ is the mean specific intensity (radiance) at the surface, which is half the specific intensity at the surface (since the radiance of the sky above the surface is zero). Thus

$$J_\nu(0) = \tfrac{1}{2} I_\nu(0) = F_\nu/(2\pi) \qquad 5.9.11$$

Therefore $$J_\nu = \frac{F_\nu}{2\pi}\left(1 + \tfrac{3}{2}\tau(\nu)\right) \qquad 5.9.12$$

This shows, to this degree of approximation (which includes the approximation that the radiation in the atmosphere is isotropic - which can be the case exactly only at the centre of the star) how the mean specific intensity increases with optical depth.

Let T be the temperature at optical depth τ.

Let T_0 be the surface temperature.

Let T_{eff} be the *effective temperature*, defined by $F(0) = \sigma T_{\text{eff}}^4$,

We also have $\pi J = \sigma T^4$ *and* $\pi J(0) = \sigma T_0^4 = \tfrac{1}{2} F$.
From these we find the following relations between these temperatures:

$$T^4 = \left(1 + \tfrac{3}{2}\tau\right) T_0^4 = \tfrac{1}{2}\left(1 + \tfrac{3}{2}\tau\right) T_{\text{eff}}^4 \qquad 5.9.13$$

$$T_0^4 = \frac{2}{2 + 3\tau} T^4 = \frac{1}{2} T_{\text{eff}}^4 \qquad 5.9.14$$

$$T_{\text{eff}}^4 = \frac{4}{2 + 3\tau} T^4 = 2T_0^4 \qquad 5.9.15$$

Note also that $T = T_{\text{eff}}$ at $\tau = 2/3$, and $T = T_0$ at $\tau = 0$.