CHAPTER 9
Bounds of a Quadric Surface

9.1 *An Easy Ellipsoid*

Consider the quadric surface

$$248x^2+179y^2+342z^2+157yz+129zx+15xy-3600=0 \qquad 9.1.1$$

The determinant Δ_3 is not zero, so it is a central quadric. There are no terms in *x* or *y* or *z*, so the centre is at the origin. In fact this is one of the quadrics discussed in Section 5.3 of Chapter 5, where we determined that it is an ellipsoid, and we also determned the orientation of its axes. Being an ellipsoid, it is bounded in *x*, *y* and *z*. We'll take these bounds to be

$$x = \pm\alpha, \quad y = \pm\beta, \quad z = \pm\gamma \qquad 9.1.2$$

Let us first see if we can find γ. For convenience,we'll work in algebra rather than in numbers, so we'll write equation 9.1.1 as

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+d=0 \qquad 9.1.3$$

The plane $z = \gamma$ intersects the quadric surface 9.1.3 in the *conic section*

$$ax^2+2hxy+by^2+2g\gamma x+2f\gamma y+(c\gamma^2+d)=0. \qquad 9.1.4$$

If γ is to be an upper or lower bound of the ellipsoid, the conic section represented by equation 9.1.4 must be a *point*. We refer now to

http://astrowww.phys.uvic.ca/~tatum/celmechs/celm2.pdf

where we find, in Section 2.7, in the tabular key on page 53, the conditions for a conic section to be a point, namely that $ab > h^2$ (this is already satisfied by the coefficients in equation 9.1.1) and $\Delta = 0$, which in the present context, is

$$\begin{vmatrix} a & h & g\gamma \\ h & b & f\gamma \\ g\gamma & f\gamma & c\gamma^2+d \end{vmatrix} = 0\,. \qquad 9.1.5$$

This results, if my algebra is correct, in

$$\gamma^2 = \frac{d(h^2-ab)}{abc+2fgh-af^2-bg^2-ch^2} = 0. \qquad 9.1.6$$

In the notation of the celm2 website cited above, this is

$$\gamma \;=\; \pm\sqrt{-d\bar{c}/\Delta}\,, \qquad 9.1.7$$

although some readers may prefer the explicit notation of equation 9.1.6. In the example of equation 9.1.1, d is negative, and so, mercifully, the argument of the square root in equation 9.1.7 is positive. I find, if my arithmetic is correct, for equation 9.1.1 the upper and lower bounds for z are $\underline{\gamma \;=\; \pm 3.5086}$.

The reader might like to find α and β by the same method. I haven't tried it, but I hope the results turn out to be $\alpha \;=\; \pm\sqrt{-d\bar{a}/\Delta}$ and $\beta \;=\; \pm\sqrt{-d\bar{b}/\Delta}$.

This example was fairly painless, because we started with a central quadric in which the centre was at the origin, there being no terms in *x*, *y* or *z*. Now let us move on to

9.2 *Three More Difficult Examples*

A. We'll start first with

$$248x^2 + 179y^2 + 342z^2 + 157yz + 129zx + 15xy - 1166x - 1261y - 1413z + 164 = 0 \qquad 9.2.1$$

It is a central quadric, because Δ_3 is not zero.

This is more difficult than the example in Section 9.1, not because of the large numbers (we'll be working in algebra, and the computer will deal with the numbers), but because there are terms in *x*, *y* and *z*; the centre of the quadric surface is not at the origin of coordinates. I suppose the easiest thing to do might be to translate the coordinate axes so the the centre of the figure *does* coincide with the new origin of coorcinates, and proceed from there. But let's not do that.

Let's suppose that the bounds of the surface are at

$$x \;=\; \alpha_1\,,\alpha_2 \;\; y \;=\; \beta_1\,,\beta_2 \quad z \;=\; \gamma_1\,,\gamma_2 \qquad 9.2.2$$

and we'll work in algebra rather than in numbers, so we'll write equation 9.2.1 as

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0. \qquad 9.2.3$$

in which

$a = 248 \;\; b = 179 \;\; c = 342 \;\; f = 78.5 \;\; g = 64.5 \;\; h = 7.5$
$u = -583 \;\; v = -630.5 \;\; w = -706.5 \;\; d = 164$

The plane $z = \gamma$ intersects the quadric surface 9.2.3 in the *conic section*

$$ax^2 + 2hxy + by^2 + 2(g\gamma + u)x + 2(f\gamma + v)y \; + (c\gamma^2 + 2w\gamma + d) = 0. \qquad 9.2.4$$

As explained in Section 9.1, if γ is to be an upper or lower bound of the ellipsoid, the conic section represented by equation 9.1.4 must be a *point*, and the conditions for this are that $ab > h^2$ and $\Delta = 0$. The first condition is already satisfied by equation 9.2.1. In the present context, the equation $\Delta = 0$ is

$$\begin{vmatrix} a & h & g\gamma + u \\ h & b & f\gamma + v \\ g\gamma + u & f\gamma + v & c\gamma^2 + 2w\gamma + d \end{vmatrix} = 0\,. \qquad 9.2.5$$

This is a quadratic equation in γ. If my algebra is correct, it comes to

$$[abc + 2fgh - af^2 - bg^2 - ch^2]\gamma^2 + 2[u(hf - bg) + v(gh - af) + w(ab - h^2)]\gamma$$

$$+\; abd - 2huv - av^2 - bu^2 - dh^2 = 0. \qquad 9.2.6$$

For the quadric surface described by equation 9.2.1, this results in

$$\underline{\gamma_1 = -2.6277 \qquad \gamma_2 = +4.6277} \qquad 9.2.7$$

To find the equation for the bounds of x and y (α and β), we can proceed either by the same method as for the bounds of z (γ), or by cyclic permutation of equation 9.2.6, or by both if we want to check the correctness of our algebra. We should obtain

$$[abc + 2fgh - af^2 - bg^2 - ch^2]\alpha^2 + 2[v(fg - ch) + w(hf - bg) + u(bc - f^2)]\alpha$$

$$+\; bcd - 2fvw - bw^2 - cv^2 - df^2 = 0 \qquad 9.2.8$$

$$[abc + 2fgh - af^2 - bg^2 - ch^2]\beta^2 + 2[w(gh - af) + u(fg - ch) + v(ca - g^2)]\beta$$

$$+\; cad - 2gwu - cu^2 - aw^2 - dg^2 = 0 \qquad 9.2.9$$

For the data of equation 9.2.1, these result in

$$\underline{\alpha_1 = -3.1063 \qquad \alpha_2 = +7.1063 \qquad \beta_1 = -0.3405 \qquad \beta_2 = +6.3405} \qquad 9.2.10$$

The ellipsoid lies within the rectangular parallelepiped (box) defined by the numbers given in equations 9.2.7 and 9.2 10. We have a bonus, in that the centre of the quadric surface must be midway between opposite faces of the box. From this, we find that the centre of the quadric surface is at (2, 3, 1).

B. Now let's look at

$$3x^2 + y^2 - 4z^2 + yz - 5zx + 9xy - 2x - 6y + 5z + 4 = 0. \qquad 9.2.11$$

Δ_3 is not zero, so it is a central quadric.

As in the previous example (**A**) we'll suppose that the bounds of the surface are at

$$x = \alpha_1, \alpha_2 \quad y = \beta, \beta_2 \quad z = \gamma_1, \gamma_2 \qquad 9.2.12$$

and we'll work in algebra rather than in numbers, so we'll write equation 9.2.11 as

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0. \qquad 9.2.13$$

in which

$a = 3 \;\; b = 1 \;\; c = -4 \;\; f = 0.5 \;\; g = -2.5 \;\; h = 4.5$
$u = -1 \;\; v = -3 \;\; w = 2.5 \;\; d = 4$

The plane $z = \gamma$ intersects the quadric surface 9.2.13 in the *conic section*

$$ax^2 + 2hxy + by^2 + 2(g\gamma + u)x + 2(f\gamma + v)y + (c\gamma^2 + 2w\gamma + d) = 0. \qquad 9.2.14$$

If this is to be a point, we must have $ab > h^2$ and $\Delta = 0$. But for equation 9.2.11, ab is *not* greater than h^2. There is no plane of the form $z = \gamma$ that touches the quadric surface at a point. The quadric surface 9.2.11 is therefore not an ellipsoid; it must be a hyperboloid.

The plane $x = \alpha$ intersects the quadric surface 9.2.13 in the conic section

$$by^2 + 2fyz + cz^2 + 2(h\alpha + v)y + 2(ga + w)z + (a\alpha^2 + 2u\alpha + d) = 0. \qquad 9.2.15$$

If this is to be a point, we must have $bc > f^2$ and $\Delta = 0$. But bc is *not* greater than f^2. There are no planes of the form $z = \gamma$ or $x = \alpha$ that touch the surface at a point. The surface is beginning to look very much like a hyperboloid of one sheet.

The plane $x = \alpha$ intersects the quadric surface 9.2.13 in the conic section

$$cz^2 + 2gzx + ax^2 + 2(f\beta + w)z + 2(hb + u)x + (b\beta^2 + 2v\beta + d) = 0. \qquad 9.2.16$$

If this is to be a point, we must have $ca > g^2$ and $\Delta = 0$. But ca is *not* greater than g^2. There are no planes of the form $z = \gamma$ or $x = \alpha$ or $y = \beta$ that touch the surface at a point. This confirms our suspicions that the surface represented by equation 9.2.11 is indeed a hyperboloid of one sheet.

C. You can probably guess what surface is represented by

$$18x^2 + 25y^2 - 4z^2 - 39yz - 3zx - 13xy + 209x - 126y + 71z + 105 = 0 \qquad 9.2.17$$

The constants are

$a = 18 \;\; b = 25 \;\; c = -4 \;\; f = -18.5 \;\; g = -1.5 \;\; h = -6.5$
$u = 104.5 \;\; v = -63 \;\; w = 30.5 \;\; d = 105$

Δ_3 is not zero, so it is a central quadric - and you have probably aready guessed which one!

We are by now accustomed to the conditions:
If a plane of the form $x = \alpha$ is to touch the surface at a point, we must have $bc > f^2$.
If a plane of the form $y = \beta$ is to touch the surface at a point, we must have $ca > g^2$.
If a plane of the form $z = \gamma$ is to touch the surface at a point, we must have $ab > h^2$.

We see that only the last of these three conditions is satisfied. No planes of the form $x = \alpha$ or $y = \beta$ touch the surface at a point. Planes of the form $z = \gamma$ touch the surface at two points given by the solution of the quadratic equation 9.2.6. I make these solutions

$$\underline{\gamma_1 = -2.6277 \qquad \gamma_2 = +4.6277} \qquad 9.2.18$$

The surface must be, as we suspected, a hyperboloid of two sheets. The z-coordinate of the centre must be halfway between these two planes, at $z = 1$.

9.3 *Bounds of a paraboloid*

Imagine an elliptic paraboloid whose axis has some random orientation. One can imagine its nose snuggled neatly into the corner of three orthogonal planes of the forms $x = \alpha$, $y = \beta$, $z = \gamma$, touching each at a single point.

Consider the quadric surface given by

$$\begin{array}{llllll} a = 13 & b = 7 & c = 79/27 & f = -4 & g = 3 & h = -8 \\ u = 9 & v = -5 & w = -4 & d = 6 & & \end{array}$$

which we found in Chapter 7 to be an elliptic paraboloid. They satisfy the conditions $bc > f^2$, $ca > g^2$, $ab > h^2$. We have to solve the quadratic equations 9.2.6, 9.2.8, 9.2.9. This at first sight looks discouraging, because we believe the paraboloid will touch each plane at only one point, whereas a quadratic equation has two points. However, we note that the coefficients of α^2 , β^2 and γ^2 in each of the equations is merely Δ_3, which, for a noncentral quadric surface is zero, so that each of the equations is only linear, having just one solution, which is exactly as expected. We quickly find that the planes that bound the elliptic paraboloid are $x = 0.0143$, $y = -0.1773$, $z = -4.8658$.

We can quickly dismiss the surface given by

$$a = 9 \quad b = 4 \quad c = 15 \quad f = 16 \quad g = -12 \quad h = -10$$
$$u = -3 \quad v = 6 \quad w = -2 \quad d = 3$$

which we know from Chapter 7 is a hyperbolic paraboloid, because $bc < f^2$, $ca < g^2$, $ab < h^2$, so, as expected, no planes touch the surface at a single point.