CHAPTER 8
General Quadratic Equation, Part IV
Cylinders and Other Possibilities

8.1 *Cylinders*

Consider

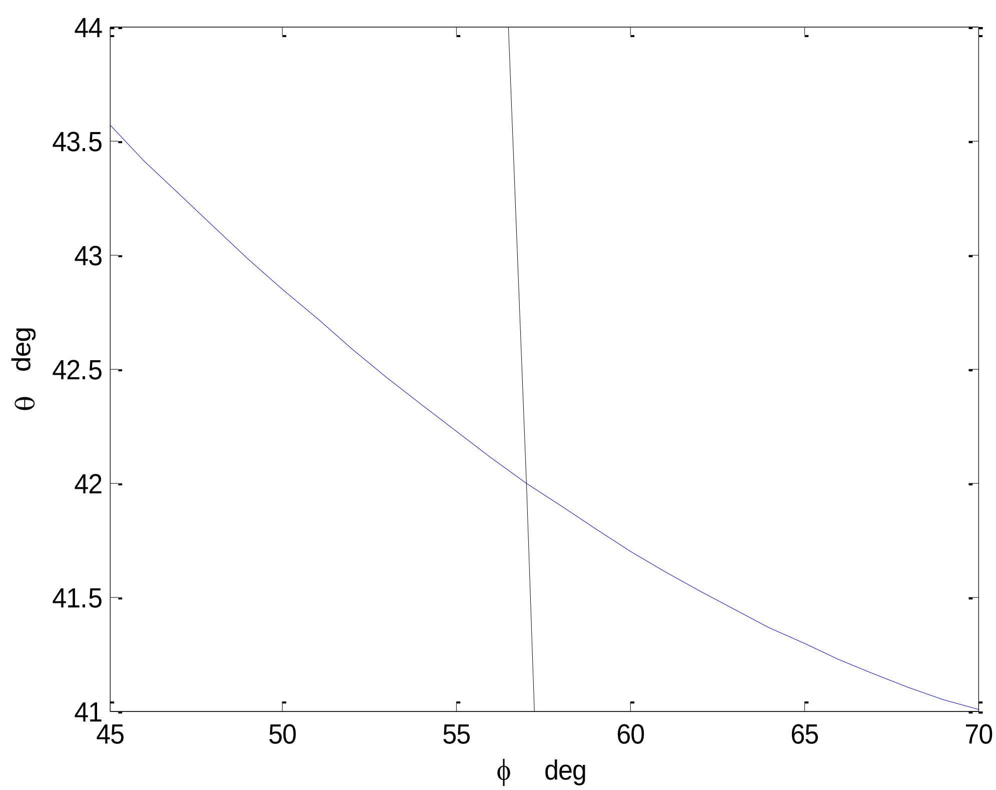
Consider $$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0 \qquad 8.1.1$$

with

a	b	c
f	g	h
u	v	w
d		

10.818990563626	6.449310193500	7.731699242874
-5.029042338510	-5.038095503461	-1.616991301910
16.303884219999	-28.140495220058	16.349186160503
71.000000000000		

I apologize for the long numbers – I worked backwards from a simple solution. However, once you have keyed them into your computer, you never need look at them again. You will find that $\Delta_3 = 0$, so it is not a central quadric. Let us find the (θ, ϕ) of the symmetry axis, as we did in Chapter 7. We'll start by drawing graphs of θ:ϕ from the equations ***f*** = 0 and ***g*** = 0. (See the Appendix to Chapter 7.)



We see that there is a solution near $\phi = 57^\circ$, $\theta = 42^\circ$. Refinement by the methods described in the Appendix to Chapter 7 shows that the solution is exactly (or at least to 12 significant figures) $\phi = 57^\circ$, $\theta = 42^\circ$. There is an antipodal solution at $\phi = 237^\circ$, $\theta = 138^\circ$.

Now we rotate the coordinate axes through these angles (follow the procedure in Chapter 7), and we find, as expected, that, in the new coordinate system (for which we use ***Franklin Gothic Book boldface italic***) , not only are ***f*** and ***g*** both zero, but ***c*** is also zero, just as it was for the paraboloids of Chapter 7. But in this case, we find, not so expected, that ***w*** is also zero.

Indeed the equation to the surface in our new coordinate system (in which the ***z*** axis is parallel to the symmetry axis of the quadric surface) becomes

$$\mathbf{a}\mathbf{x}^2 + 2\mathbf{h}\mathbf{x}\mathbf{y} + \mathbf{b}\mathbf{y}^2 + 2\mathbf{u}\mathbf{x} + 2\mathbf{v}\mathbf{y} + \mathbf{d} = 0 \qquad 8.1.2$$

with $\mathbf{a} = 14$ $\mathbf{h} = -2$ $\mathbf{b} = 11$ $\mathbf{u} = -22$ $\mathbf{v} = -29$ $\mathbf{d} = 71$

There is no ***z*** in the equation at all! In two dimensions, we recognize this as a conic section, and with the above values of the constants, it is an ellipse with centre (2 , 3). In three dimensions, however, it is a **cylinder** of elliptic cross-section - a cross-section that is independent of ***z***.

If we had arrived at an equation similar to equation 8.1.2, but with different values of the constants, we might have arrived at a cylinder of hyperbolic or parabolic cross-section (perhaps stretching the meaning of what we usually think of as a cylinder). Lovers of the conic sections will recognize that there are other possibilities for equation 8.1.2 (see astrowww.phys.uvic.ca/~tatum/celmechs/celm2.pdf, especially the table on page 50). It could represent, in 2-space, two straight lines, which corresponds, in 3-space, to two planes. Or it could represent, in 2-space, a single point, which corresponds, in 3-space, to a straight line. Or it could represent nothing at all!

8.2 *A Straight Line*

Consider

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \qquad 8.2.1$$

with

a	b	c
f	g	h
u	v	w
d		

2.101846382587	0.194785295875	0.703368321538
-0.042895630528	-0.954028098231	-0.335823005545
-0.238754038366	-0.489518322054	0.838670567945
2.000000000000		

$\Delta_3 = 0$, so it is not a central quadric. Is it a paraboloid? Is it a cylinder? We shall see.

Let us erect a coordinate system ***xyz*** ("***Franklin Gothic Book boldface italic***") such that the ***z*** axis is parallel to the symmetry axis of the figure represented by equation 8.2.1, in a manner similar to how we did this in Chapter 7. We shall find that the spherical angles of the ***z*** axis, referred to the *xyz* coordinate axes are $\theta = 33^\circ$, $\phi = 64^\circ$. These angles satisfy not only $\boldsymbol{f} = 0$, $\boldsymbol{g} = 0$ and $\boldsymbol{c} = 0$ (as in the examples in Chapter 7) but they also satisfy $\boldsymbol{w} = 0$. We find that the equation to the quadric, when referred to the ***xyz*** system, becomes merely

$$\boldsymbol{x}^2 - 2\boldsymbol{xy} + 2\boldsymbol{y}^2 - 2\boldsymbol{x} + 2 = 0 \qquad 8.2.2$$

In the ***xy*** plane, this is satisfied only by the single point (2 , 1). In three dimensions, equation 8.2.2 represent straight line perpendicular to the ***xy*** plane. Thus we find that equation 8.2.1, when referred to the *xyz* coordinate axes, is a straight line with spherical angles $\theta = 33^\circ$, $\phi = 64^\circ$.

An alternative treatment follows.

The figure represented by equation 8.2.1 intersects the coordinate axes where:

x-axis:	$ax^2 + 2ux + d = 0$	$u^2 < ad$	No real points	8.2.3
y-axis:	$by^2 + 2vy + d = 0$	$v^2 < bd$	No real points	8.2.4
z-axis:	$cz^2 + 2wz + d = 0$	$w^2 < cd$	No real points	8.2.5

It intersects the coordinate planes in the following conic sections:

yz-plane: $by^2 + 2fyz + cz^2 + 2vy + 2wz + d = 0$ 8.2.6

zx-plane: $cz^2 + 2gzx + ax^2 + 2wz + 2ux + d = 0$ 8.2.7

xy-plane: $ax^2 + 2hxy + by^2 + 2ux + 2vy + d = 0$ 8.2.8

It will require some knowledge of conic sections (see for example astrowww.phys.uvic.ca/~tatum/celmechs/celm2.pdf especially page 50)
to determine what sort of conic sections these are. With the constants given below equation 8.2.1, we find that:

Equation 8.2.6 represents a single point P in the *yz* plane, whose coordinates are given by

$$y = \frac{fw - cv}{bc - f^2}, \qquad z = \frac{fv - bw}{bc - f^2}, \qquad x = 0 \qquad 8.2.9$$

Equation 8.2.7 represents a single point Q in the *zx* plane, whose coordinates are given by

$$z = \frac{gu - aw}{ca - g^2}, \qquad x = \frac{gw - cu}{ca - g^2}, \qquad y = 0 \qquad 8.3.10$$

Equation 8.2.8 represents a single point R in the *xy* plane, whose coordinates are given by

$$x = \frac{hv - bu}{ab - h^2}, \qquad y = \frac{hu - av}{ab - h^2}, \qquad z = 0 \qquad 8.3.11$$

Numerically these are:

P:	$(x,\ y,\ z) =$	0.000000000000	2.281172032705	-1.053243702588
Q:	$(x,\ y,\ z) =$	-1.112601940475	0.000000000000	-2.701463832136
R:	$(x,\ y,\ z) =$	0.710973592838	3.738883921362	0.000000000000

From equations 1.8 to 1.10 of Chapter 1 we can readily calculate the direction cosines of the lines PQ, PR, QR, and we find hence that PQR is a straight line. That is, we have discovered that equation 8.2.1, with the constants given, represent a single **straight line.** Its direction cosines are

$$l = \frac{x_P - x_Q}{s}, \quad m = \frac{y_P - y_Q}{s}, \quad n = \frac{z_P - z_Q}{s} \quad , \qquad 8.3.12$$

where $$s = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2}\,. \qquad 8.3\ 13$$

That is,

l	m	n	
0.367648978648	0.753792113276	0.544639035015	8.3.14

These are parallel to the "Franklin" coordinate system ***xyz***, and are the same as the l_3 , m_3 , n_3 of equation 7.1.12. Therefore the spherical angles of the line, referred to the *xyz* coordinates, can be calculated from

$$\sin\theta = n \qquad \cos\phi = m/n \qquad 8.3.15$$

Hence $\underline{\theta = 33^\circ,\ \phi = 64^\circ.}$

8.3 *Nothing*

Consider

$$96x^2 + 192y^2 + 30z^2 + 144yz - 48zx - 192xy + 12x - 4y + 8z + 12 = 0. \qquad 8.4$$

That is:

a	b	c
f	g	h
u	v	w
d		
96	192	30
72	–24	–96
6	–2	4
12		

This surface cuts the
x-axis where $96x^2 + 12x + 12 = 0$ i.e. nowhere
y-axis where $192y^2 - 4x + 12 = 0$ i.e. nowhere
z-axis where $30z^2 + 8z + 12 = 0$ i.e. nowhere

It cuts the
$x = 0$ plane where $192y^2 + 144yz + 30z^2 - 4y + 8z + 12 = 0$ i.e. nowhere
$y = 0$ plane where $30z^2 - 48zx + 96x^2 + 8z + 12x + 12 = 0$ i.e. nowhere
$z = 0$ plane where $96x^2 - 192xy + 192y^2 + 12x - 4y + 12 = 0$ i.e. nowhere

This might be an ellipsoid - except that $\Delta_3 = 0$, so it cannot be a central quadric. There is no point (x, y, z) that satisfies equation 8.4.