

CHAPTER 7
General Quadratic Equation, Part II
 $\Delta_3 = 0$: **Noncentral Surfaces**
Paraboloids

7.1 *An Elliptic Paraboloid*

Chapter 6 dealt with equations of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \quad 7.1.1$$

that could be factorized into two real linear factors, and therefore represented two planes. But consider equation 7.1.1 with

$$\begin{array}{llllll} a = 13 & b = 7 & c = 79/27 & f = -4 & g = 3 & h = -8 \\ u = 9 & v = -5 & w = -4 & d = 6 & & \end{array} \quad 7.1.2$$

You will find that $\Delta_3 = 0$, so it does not represent a central quadric surface, so it is either two planes or a paraboloid. But try as you will, you will not be able to split this into two real linear factors. It must be a paraboloid. [I apologize for the fractional value for c . I chose the other coefficients quite arbitrarily as simple integers. However, to ensure that the surface is not a central surface, I was constrained by the requirement that $\Delta_3 = 0$, which then determined the value of c .]

We'll note in passing, for future reference, that $\text{Tr } \Delta_3 = 22.\overline{925}$. [$\text{Tr } \Delta_3$ means the *trace* of Δ_3 , or the sum of its diagonal elements, $a + b + c$.]

The surface represented by equation 7.1.1 intersects the coordinate axes where:

$$\text{x-axis: } ax^2 + 2ux + d = 0 \quad u^2 > ad \quad \text{Two real points} \quad 7.1.3$$

$$\text{y-axis: } by^2 + 2vy + d = 0 \quad v^2 < bd \quad \text{No real points} \quad 7.1.4$$

$$\text{z-axis: } cz^2 + 2wz + d = 0 \quad w^2 < cd \quad \text{No real points} \quad 7.1.5$$

It intersects the coordinate planes where:

$$\text{yz-plane: } by^2 + 2fyz + cz^2 + 2vy + 2wz + d = 0 \quad f^2 < bc \quad \text{Ellipse} \quad 7.1.6$$

$$\text{zx-plane: } cz^2 + 2gzx + ax^2 + 2wz + 2ux + d = 0 \quad g^2 < ca \quad \text{Ellipse} \quad 7.1.7$$

$$\text{xy-plane: } ax^2 + 2hxy + by^2 + 2ux + 2vy + d = 0 \quad h^2 < ab \quad \text{Ellipse} \quad 7.1.8$$

It is evident by now that we must be dealing with an elliptic paraboloid.

In this Chapter, we shall often be rotating or translating the axes of coordinates, so, in order to distinguish between the various coordinate systems, I am going to use different

fonts for the different systems. Thus, in my computer, equation 7.1.1 is set in *Times New Roman Italic*, whereas equation 7.1.9 below is set in ***Franklin Gothic Book Bold Italic***.

In Chapter 1, we discussed the rotation of coordinate systems. I reproduce figure I.5 here:

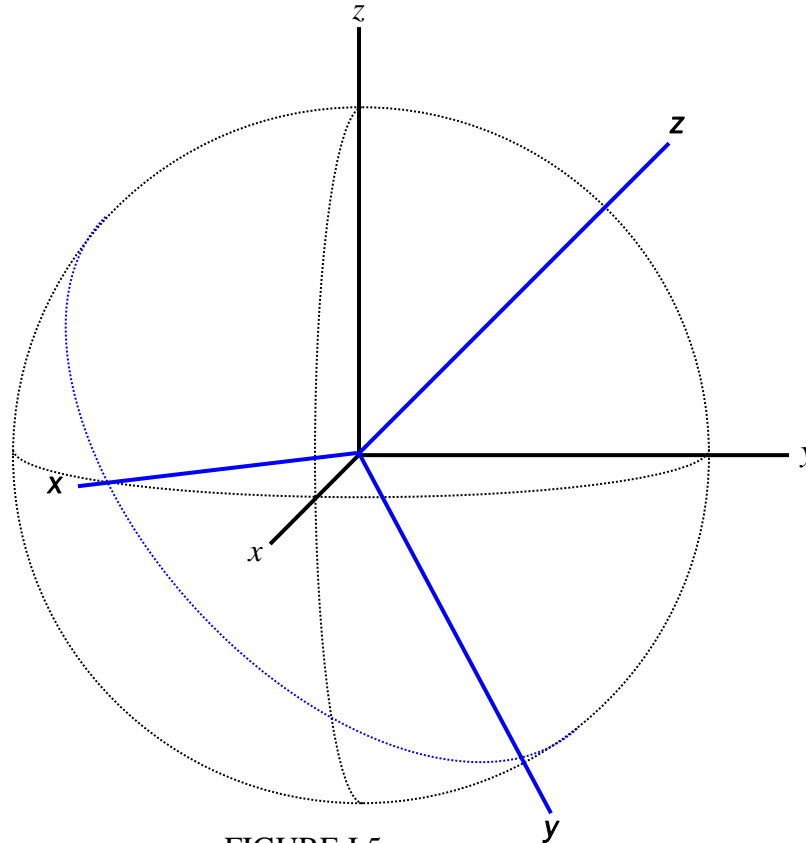


FIGURE I.5

We have a black coordinate system xyz , set in *Times New Roman italic* (and referred to as the “Roman” coordinate system), and a blue system ***xyz***, set in ***Franklin Gothic Book italic boldface***, and referred to as the “Franklin” system. I take the direction cosines of the ***x***, ***y***, ***z*** axes (relative to the xyz system) to be, respectively (l_1, m_1, n_1) , (l_2, m_2, n_2) , (l_3, m_3, n_3) , and the spherical coordinates of the ***z*** axis to be (θ, ϕ) .

A point in space can be represented by its Roman coordinates (x, y, z) or by its Franklin coordinates ***(x, y, z)***, and it was shown in Chapter 1 that these coordinates are related through:

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sin \phi & -\cos \phi & 0 \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad 1.2.13$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \sin \phi & \cos \theta \cos \phi & \sin \theta \cos \phi \\ -\cos \phi & \cos \theta \sin \phi & \sin \theta \sin \phi \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} \quad 1.2.14$$

For example, if we have a quadric surface represented in Roman coordinates by

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0, \quad 7.1.1$$

when referred to Franklin coordinates, it becomes

$$\mathbf{a}x^2 + \mathbf{b}y^2 + \mathbf{c}z^2 + 2\mathbf{f}yz + 2\mathbf{g}zx + 2\mathbf{h}xy + 2\mathbf{u}x + 2\mathbf{v}y + 2\mathbf{w}z + \mathbf{d} = 0 \quad 7.1.9$$

in which

$$\begin{aligned} \mathbf{a} &= al_1^2 + bm_1^2 + cn_1^2 + 2(fm_1n_1 + gn_1l_1 + hl_1m_1) \\ \mathbf{b} &= al_2^2 + bm_2^2 + cn_2^2 + 2(fm_2n_2 + gn_2l_2 + hl_2m_2) \\ \mathbf{c} &= al_3^2 + bm_3^2 + cn_3^2 + 2(fm_3n_3 + gn_3l_3 + hl_3m_3) \\ \mathbf{f} &= al_2l_3 + bm_2m_3 + cn_2n_3 + f(m_2n_3 + m_3n_2) + g(n_2l_3 + n_3l_2) + h(l_2m_3 + l_3m_2) \\ \mathbf{g} &= al_3l_1 + bm_3m_1 + cn_3n_1 + f(m_3n_1 + m_1n_3) + g(n_3l_1 + n_1l_3) + h(l_3m_1 + l_1m_3) \\ \mathbf{h} &= al_1l_2 + bm_1m_2 + cn_1n_2 + f(m_1n_2 + m_2n_1) + g(n_1l_2 + n_2l_1) + h(l_1m_2 + l_2m_1) \\ \mathbf{u} &= ul_1 + vm_1 + wn_1 \\ \mathbf{v} &= ul_2 + vm_2 + wn_2 \\ \mathbf{w} &= ul_3 + vm_3 + wn_3 \\ \mathbf{d} &= d \end{aligned}$$

7.1.10a - j

Now, according to equation 4.1.7 of Chapter 4, an equation of the form

$$\mathbf{a}x^2 + 2\mathbf{h}xy + \mathbf{b}y^2 + 2\mathbf{u}x + 2\mathbf{v}y + 2\mathbf{w}z + \mathbf{d} = 0 \quad \text{which was 4.1.7, is now} \quad 7.1.11$$

(with no terms in \mathbf{z}^2 , \mathbf{yz} or \mathbf{zx}) represents a paraboloid in which the \mathbf{z} axis is parallel to the symmetry axis of the paraboloid. We are going to try rotating the axes of coordinates in such a manner that equation 7.1.1 takes the form of 7.1.11. We want to choose θ and ϕ so that $\mathbf{f} = \mathbf{g} = \mathbf{c} = 0$. In order not to interrupt the flow of thought, I give the solutions here for our numerical example:

$$\theta = 48^\circ.091 \ 694 \ 731 \ 9$$

$$\phi = 68^\circ.552 \ 263 \ 672 \ 9$$

(The method of solution to arrive at these angles is given in the Appendix to this Chapter.)

These angles give the orientation of the symmetry axis of the paraboloid referred to the original xyz coordinate system.

There is another solution, namely

$$\theta = 131^\circ.908\ 503\ 268\ 1 \quad \phi = 355^\circ.552\ 263\ 672\ 9$$

This is merely the same orientation, in the diametrically opposite direction.

Having determined θ and ϕ , we can now (by using equations 1.2.13 and 7.1.10) calculate all the constants in equation 7.1.9. These are:

a	b	c
f	g	h
u	v	w
d		
17.643093922652	5.282832003274	0.000000000000
0.000000000000	0.000000000000	-5.717253072310
10.205025303078	2.066532181884	-3.686041685165
6.000000000000		

Thus we have found a set of coordinate axes such that the equation to the paraboloid is of the form of equation 7.1.11 which, we have noted, is a paraboloid with its symmetry axis parallel to the \mathbf{z} -axis of the coordinate system.

We note with satisfaction that $\mathbf{a} + \mathbf{b} + \mathbf{c} = a + b + c = 22.\overline{925}$ (the trace of a matrix is unaltered by an orthogonal transformation) and also that $\Delta_3 = 0$ (the surface is still a paraboloid).

Those who are familiar with the conic sections are now on familiar ground. They will note that for any constant \mathbf{z} , equation 7.1.11 is the equation to a conic section. They will also note that $h^2 < \mathbf{ab}$, so that the cross-section of the paraboloid at any constant \mathbf{z} is an ellipse.

The next step is to rotate the coordinate system so as to get rid of the term in \mathbf{xy} , so that the axes of the elliptical cross-section are parallel to the axes of a new coordinate system. We choose a new coordinate system xyz (*Bookman Old Style Italic* on my computer) such that the new x -axis makes an angle χ with the \mathbf{x} axis:

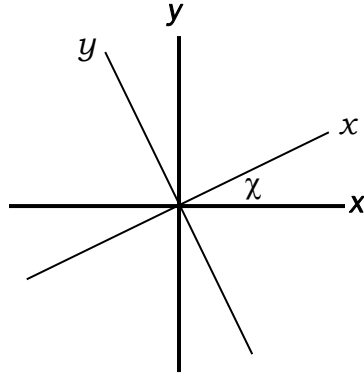


FIGURE VII.2

so that

$$\begin{aligned}x &= x' \cos \chi - y' \sin \chi \\y &= x' \sin \chi + y' \cos \chi\end{aligned}\tag{7.1.12}$$

On substituting these into equation 7.1.11, we find that the equation to the paraboloid now becomes

$$ax'^2 + 2h'x'y' + by'^2 + 2u'x' + 2v'y' + 2wz + d = 0,\tag{7.1.13}$$

where

$$\begin{aligned}a &= a \cos^2 \chi + b \sin^2 \chi + 2h \sin \chi \cos \chi \\b &= a \sin^2 \chi + b \cos^2 \chi - 2h \sin \chi \cos \chi \\h &= -a \sin \chi \cos \chi + b \sin \chi \cos \chi + h \cos^2 \chi - h \sin^2 \chi \\u &= u \cos \chi + v \sin \chi \\v &= -u \sin \chi + v \cos \chi \\w &= w \\d &= d.\end{aligned}\tag{7.1.14}$$

We choose χ so that $h = 0$. The coefficient h can be written conveniently as

$$h = \frac{1}{2}(b - a) \sin 2\chi + h \cos 2\chi\tag{7.1.15}$$

so we choose χ so that

$$\tan 2\chi = \frac{2h}{a - b}.\tag{7.1.16}$$

In our particular example, $\tan 2\chi = -0.925102252622$. This gives four solutions between 0 and 360 degrees, at right angles to each other, namely

$$\chi = 68^\circ.614008767109, 158^\circ.614008767109, \\ 248^\circ.614008767109, 338^\circ.614008767109.$$

These are the directions (referred to the xyz system) of the x and y axes. I shall choose the first of these solutions to be the direction of the positive x axis,

We can now calculate, using equations 7.1.14, the constants $abhuvwd$.

a	b	h
u	v	w
d		
3.043877420474	19.882048505452	17.974267952307
5.645494718445	-8.748799122121	-3.686041685165
6.000000000000		

and the equation to the surface is now

$$ax^2 + by^2 + 2ux + 2vy + 2wz + d = 0. \quad 7.1.17$$

We note with satisfaction that the trace ($a + b$) is unaltered.

$$\text{Now let } x = \mathbf{x} + x_0 \quad y = \mathbf{y} + y_0 \quad z = \mathbf{z} + z_0 \quad 7.1.18$$

so that the equation now becomes

$$a\mathbf{x}^2 + b\mathbf{y}^2 + 2u\mathbf{x} + 2v\mathbf{y} + 2w\mathbf{z} + d = 0, \quad 7.1.19$$

in which

$$\begin{aligned} \mathbf{a} &= a \\ \mathbf{b} &= b \\ \mathbf{u} &= ax_0 + u \\ \mathbf{v} &= by_0 + v \\ \mathbf{w} &= w \\ \mathbf{d} &= ax_0^2 + by_0^2 + 2ux_0 + 2vy_0 + 2wz_0 + d = 0 \end{aligned} \quad 7.1.20$$

[I have used *Lucida Bright italic bold* font for the new coordinate system.]

We choose x_0, y_0, z_0 so that $\mathbf{u}, \mathbf{v}, \mathbf{d}$ are all zero. That is

$$x_0 = -\frac{u}{a} \quad y_0 = -\frac{v}{b} \quad z_0 = \frac{-(ax_0^2 + by_0^2 + 2ux_0 + 2vy_0 + d)}{2w} \quad 7.1.21$$

In our example, these are:

x_0	y_0	z_0
-1.854705015541	0.440035095967	-1.128650560885

and the equation to the surface is now

$$ax^2 + by^2 + 2wz = 0, \quad 7.1.22$$

with

a	b	w
3.043877420474	19.882048505452	-3.686041685165

This can be written in standard form as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{h} \quad 7.1.23$$

with

a	b	h
0.573173918874	0.224269097566	0.271293730623

[I have used *Harrington Italic* font for the new constants.]

The cross-section of the surface in any plane perpendicular to the Z axis is an ellipse of

eccentricity $\sqrt{1 - b^2/a^2} = \sqrt{1 - a/b} = \underline{0.920273453562}$.

7.2 A HyperbolicParaboloid

Consider

$$9x^2 + 4y^2 + 15z^2 + 32yz - 24zx - 20xy - 6x + 12y - 4z + 3 = 0 \quad 7.2.1$$

$$\begin{aligned} a = 9 \quad b = 4 \quad c = 15 \quad f = 16 \quad g = -12 \quad h = -10 \\ u = -3 \quad v = 6 \quad w = -2 \quad d = 3 \end{aligned} \quad 7.2.2$$

$\Delta_3 = 0$, so it is not a central quadric.

It intersects the coordinate axes where:

$$\begin{aligned} x\text{-axis:} \quad ax^2 + 2ux + d = 0 \quad u^2 < ad \quad \text{No real points} \\ y\text{-axis:} \quad by^2 + 2vy + d = 0 \quad v^2 > bd \quad \text{Two real points} \\ z\text{-axis:} \quad cz^2 + 2wz + d = 0 \quad w^2 < cd \quad \text{No real points} \end{aligned} \quad 7.2.3$$

Two planes cannot behave like this.

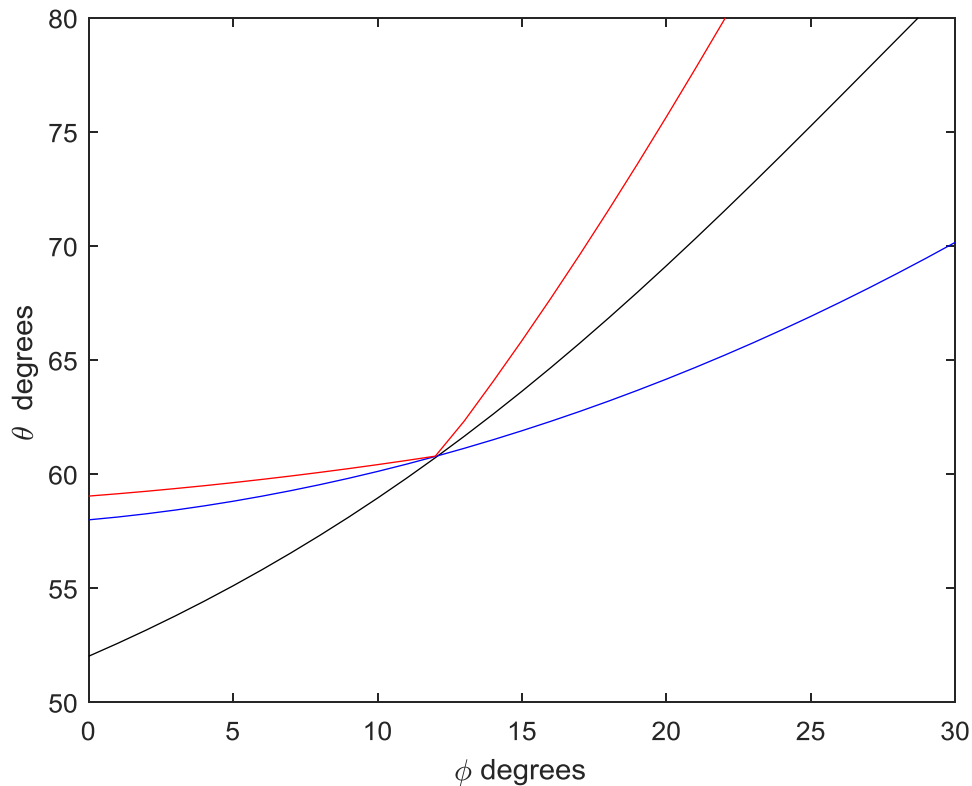
It intersects the coordinate planes where:

$$\begin{aligned} yz\text{-plane:} \quad by^2 + 2fyz + cz^2 + 2vy + 2wz + d = 0 \\ zx\text{-plane:} \quad cz^2 + 2gzx + ax^2 + 2wz + 2ux + d = 0 \\ xy\text{-plane:} \quad ax^2 + 2hxy + by^2 + 2ux + 2vy + d = 0 \end{aligned} \quad 7.2.4$$

These are all conic sections, and lovers of the conic sections will note that $f^2 > bc$, $g^2 > ca$, $h^2 > ab$, so that each of these sections is a hyperbola. We are evidently dealing with a hyperbolic paraboloid.

Suppose that the spherical coordinates of the symmetry axis of the paraboloid are (θ, ϕ) referred to the xyz coordinate axes. Now, following the methods described in Section 7.1 for the elliptical case and the Appendix to this Chapter, we have to transform to an \mathbf{xyz} coordinate system, such that the symmetry axis of the paraboloid is parallel to the \mathbf{z} coordinate axis, and so determine (θ, ϕ) .

We start by drawing graphs of $\mathbf{f}(\phi)$, $\mathbf{g}(\phi)$ and $\mathbf{c}(\phi)$ to see for what values of ϕ and θ are these three functions equal. The graphs are shown below, with \mathbf{f} , \mathbf{g} , \mathbf{c} in black, blue and red respectively.



We see that there is a solution near $\theta = 60^\circ.8$, $\phi = 12^\circ.1$, and we leave it to the reader, using one or both of the methods described in the Appendix, to refine this to

$$\theta = 60^\circ.8059760832, \phi = 12^\circ.0947570770$$

As explained in the Appendix as well as in Section 7.1 of the Chapter, there will be two solutions, namely (θ, ϕ) and $(180^\circ - \theta, 180^\circ + \phi)$.

Make the transform to *Franklin* coordinates, in which the z axis is parallel with the symmetry axis of the paraboloid. I make it

$$ax^2 + 2hxy + by^2 + 2ux + 2vy + 2wz + d = 0 \quad 7.2.5$$

with

$$\begin{array}{ccc} \mathbf{a} & \mathbf{b} & \mathbf{h} \\ \mathbf{u} & \mathbf{v} & \mathbf{w} \end{array}$$

$$\begin{array}{ccc} 19.682926829273 & 8.317073170727 & -20.801546640890 \\ 0.642167469770 & 6.621119183913 & -2.951000001109 \\ 3.000000000000 & & \end{array}$$

In any plane $\mathbf{z} = \text{constant}$, this is a hyperbola. The trace is 28 in both the original *Roman* and the new *Franklin* coordinates.

To move to Bookman coordinates (in which the x and y axes are parallel to the principal axes of the hyperbolic cross-section of the paraboloid), we must rotate the axes through χ , such that $\tan 2\chi = \frac{2h}{a-b}$. The four principal solutions are

$$\chi = 52^\circ.640091510845, 142^\circ.640091510845, \\ 232^\circ.640091510845, 322^\circ.640091510845.$$

I choose $\chi = 232^\circ.640091510845$ for the positive x axis. (I first chose 52, but I changed my mind – as you probably will – as soon as I had calculated the Bookman coefficients.)

I find that $ax^2 + by^2 + 2ux + 2vy + 2wz + d = 0$. 7.2.6
with

a	b	d
u	v	w
35.563858652848	-7.563858652848	3.000000000000
3.203663450766	-5.726131774932	-2.438843043399

This is a hyperbola, and the trace, $a + b$, is still 28.

Next we need to find x_0 y_0 z_0 such that, in Lucida Bright coordinates (see Setion 7.1) there are no terms in \mathbf{x} \mathbf{y} \mathbf{z} . I obtain

x_0	y_0	z_0
-0.090081998189	-0.757038442644	1.444600851861

and hence $ax^2 + by^2 + 2wz = 0$ 7.2.7

where

a	b	w
35.563858652848	-7.563858652848	-2.438843043399

and hence

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{h} \quad 7.2.8$$

with

a	b	h
0.363603703205	0.167685520107	0.338868180151

The cross-section of the surface in any plane perpendicular to the Z axis is a hyperbola of eccentricity $\sqrt{1 + b^2/a^2} = 1.101219259592$.

Appendix 7A

How to solve the equations $f = g = c = 0$ for θ and ϕ .

We have to solve the equations $f = g = c = 0$ for θ and ϕ . These three equations are not independent; we can solve two of them, and then use the third as an additional check. I choose to solve $f = 0$ and $g = 0$.

After some algebra and trigonometric manipulations, these equations can be written as

$f = 0$:

$$\tan 2\theta = \frac{-2(f \sin \phi + g \cos \phi)}{a \cos^2 \phi + b \sin^2 \phi - c + h \sin 2\phi} \quad 7.A.1$$

$g = 0$:

$$\tan \theta = \frac{f \cos \phi - g \sin \phi}{\frac{1}{2}(a - b) \sin 2\phi - h \cos 2\phi} \quad 7.A.2$$

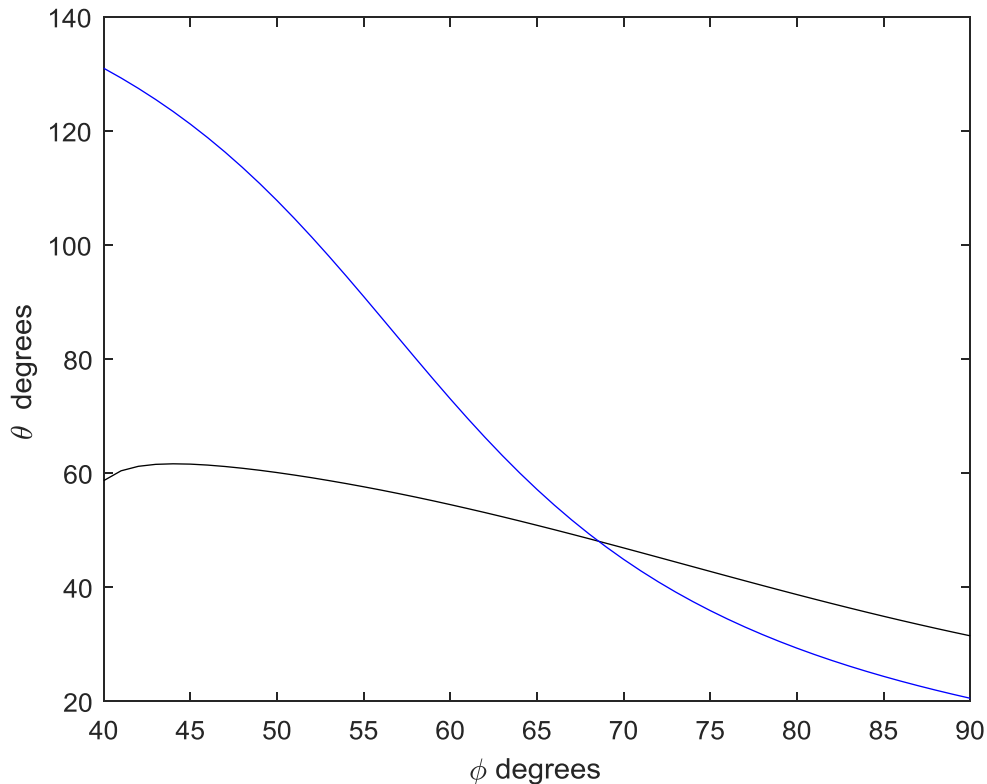
For the elliptical paraboloid in our example in Chapter 7, numerical values of the coefficients $abc fgh$ are given in equation 7.1.2. Here they are again:

$$\begin{array}{llllll} a = & 13 & b = & 7 & c = & 79/27 & f = & -4 & g = & 3 & h = & -8 \\ u = & 9 & v = & -5 & w = & -4 & d = & 6 & & & & \end{array} \quad 7.1.2$$

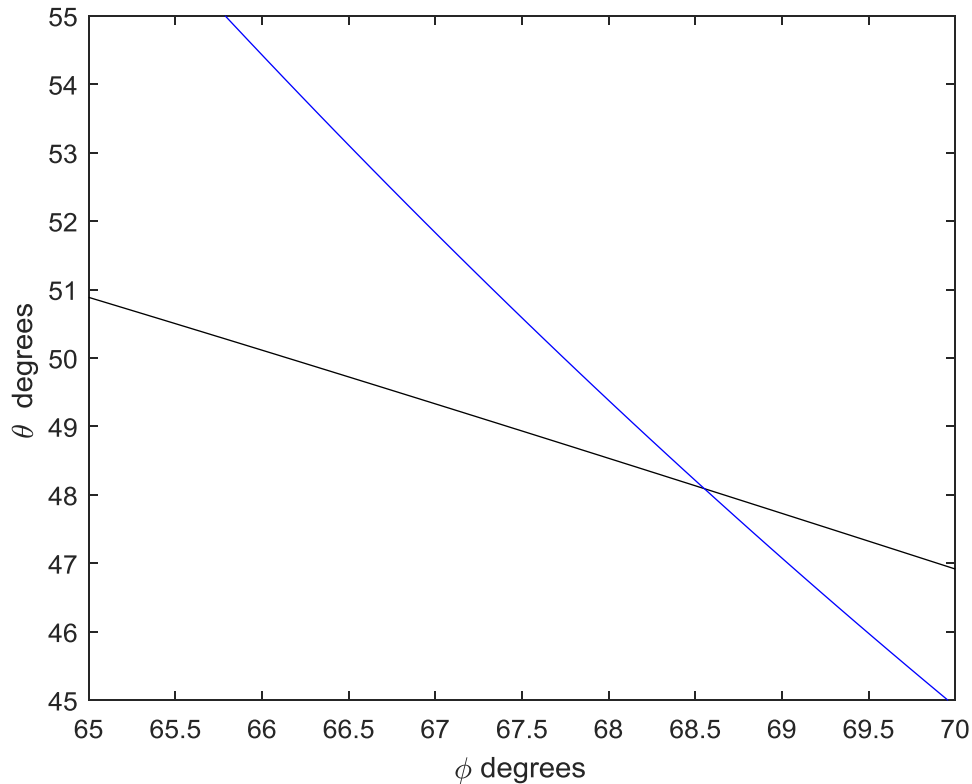
We are looking for solutions in the range $\theta = 0$ to 180 degrees, $\phi = 0$ to 360 degrees. Recall also that we are looking for a (θ, ϕ) that is going to represent the orientation of the symmetry axis of the paraboloid. There will be two solutions for (θ, ϕ) in diametrically opposite directions. That is, if one solution is (θ, ϕ) , there will be another solution $(180^\circ - \theta, 180^\circ + \phi)$ in the opposite direction. (Signs *sic*.)

I suggest two methods of solving them. The calculation may look formidable, but, once the necessary equations have been developed, they are straightforward to program into a computer, and the numerical calculation by computer is instantaneous.

In both methods we must start with a guess at the solutions. How may we make such a guess? One way is to draw graphs of $\theta : \phi$ from equations 7.A.1 and 7.A.2 (i.e. $\mathbf{f} = 0$ and $\mathbf{g} = 0$) and see where they cross (i.e. where they both have the same values of (θ, ϕ)). Here they are - $\mathbf{f} = 0$ in black and $\mathbf{g} = 0$ in blue.



We see from the graphs that there is a solution near $\theta = 48^\circ$, $\phi = 69^\circ$ (There is another solution near $\theta = 132^\circ$, $\phi = 249^\circ$.) This should be good enough for our first guess, but if we want a better guess, it is easy to ask the computer to zoom in for a closer look:



We see from these graphs that there is a solution close to $\theta = 48^\circ.1$, $\phi = 68^\circ.6$. This is not only easily good enough for a first guess, but it may be all the precision you want. However, let us proceed with a numerical calculation anyway. Two methods.

METHOD 1

$f = 0$:

$$\tan 2\theta = \frac{-2(f \sin \phi + g \cos \phi)}{a \cos^2 \phi + b \sin^2 \phi - c + h \sin 2\phi} \quad 7.A.1$$

$g = 0$:

$$\tan \theta = \frac{f \cos \phi - g \sin \phi}{\frac{1}{2}(a - b) \sin 2\phi - h \cos 2\phi} \quad 7.A.2$$

In this method we note that we can eliminate θ between the two equations, and so obtain a single equation in ϕ , which can be solved by the standard single-variable Newton-Raphson procedure. Thus, if we denote the right hand sides of equation 7.A.1 by

Q , and the right hand side of equation 7.A.2 by P , and $\tan \theta$ by t , the two equations can be written as

$$\frac{2t}{1-t^2} = Q \quad t = P \quad 7A,3,4$$

from which t is easily eliminated to form

$$F = Q(PQ + 2) - P = 0. \quad 7A.5$$

and we have to solve 7A.5 for ϕ . This can be done by Newton-Raphson iteration, for which we need F' . where the prime indicates the ϕ -derivative. On differentiating equation 7A.5, we obtain

$$F' = 2Q'(PQ + 1) + P'(Q^2 - 1) , \quad 7A.6$$

and P' and Q' are obtained by differentiation of the right hand sides of equations 7A.2 and 7A.1:

Here is a step-by-step algorithm, easily programmed for a computer

1. Read a, b, c, f, g, h
2. Guess ϕ .
3. Calculate and store $\sin \phi, \cos \phi, \sin 2\phi, \cos 2\phi$
4. Calculate P and P' thus:

$$\begin{aligned} U &= -2(f \sin \phi + g \cos \phi) \\ V &= a \cos^2 \phi + b \sin^2 \phi - c + h \sin 2\phi \\ P &= U/V \\ U' &= -2(f \cos \phi - g \sin \phi) \\ V' &= (b - a) \sin 2\phi + 2h \cos 2\phi \\ P' &= (VU' - UV')/V^2 \end{aligned}$$

5. Calculate Q and Q' thus:

$$\begin{aligned} U &= f \cos \phi - g \sin \phi \\ V &= \frac{1}{2}(a - b) \sin 2\phi - h \cos 2\phi \\ Q &= U/V \\ U' &= -f \sin \phi - g \cos \phi \\ V' &= (a - b) \cos 2\phi + 2h \sin 2\phi \\ Q' &= (VU' - UV')/V^2 \end{aligned}$$

6. Calculate F and F' :

$$F = Q(PQ + 2) - P$$

$$F' = 2Q'(PQ + 1) + P'(Q^2 - 1)$$

7. Calculate new ϕ :

$$\phi_{\text{new}} = \phi_{\text{old}} - F / F'$$

8. Calculate $\frac{|\phi_{\text{new}} - \phi_{\text{old}}|}{\phi_{\text{old}}}$

9. EITHER: Replace the old ϕ with the new, and return to step 3.

OR, if $\frac{|\phi_{\text{new}} - \phi_{\text{old}}|}{\phi_{\text{old}}}$ is as small as you wish it to be, STOP.

We already have an initial guess of $\phi = 68^\circ.6$. I obtained $\phi = 68^\circ.552263672895$ after 4 iterations. The calculation was done apparently instantaneously.

Either of equations 7A.1 or 2 yields $\theta = 48^\circ.091694731928$.

These solutions also satisfy the equation $\mathbf{c} = 0$. (This should be verified!)

The opposite direction $180^\circ - \theta$, $180^\circ + \phi$ is also a solution.

METHOD 2

We have chosen to solve

$\mathbf{f} = 0$:

$$\tan 2\theta = \frac{-2(f \cos \phi + g \sin \phi)}{a \cos^2 \phi + b \sin^2 \phi - c + h \sin 2\phi} \quad 7.A.1$$

$\mathbf{g} = 0$:

$$\tan \theta = \frac{f \cos \phi - g \sin \phi}{\frac{1}{2}(a - b) \sin 2\phi - h \cos 2\phi} \quad 7.A.2$$

by the two-variable extension of the Newton-Raphson procedure. For this, we shall need the partial derivatives of \mathbf{f} and \mathbf{g} with respect to θ and to ϕ .

We then proceed with the extended Newton-Raphson iteration described in Appendix 5A.

Here's an algorithm, easy to program.

1. Read a, b, c, f, g, h
2. Guess θ and ϕ .
3. Calculate and store $\sin \theta$, $\cos \theta$, $\tan \theta$, $\cos 2\theta$, $\tan 2\theta$, $\sin \phi$, $\cos \phi$, $\sin 2\phi$, $\cos 2\phi$
4. Calculate the function f and its derivatives. Derivatives indicated by subscripts.

$$\begin{aligned} U &= -2(f \sin \phi + g \cos \phi) \\ V &= a \cos^2 \phi + b \sin^2 \phi - c + 2h \sin 2\phi \\ f &= \tan 2\theta - U/V \end{aligned}$$

$$f_{\theta} = 2 \sec^2 2\theta$$

$$\begin{aligned} U_{\phi} &= -2(f \cos \phi - g \sin \phi) \\ V_{\phi} &= (b - a) \sin 2\phi + 4h \cos 2\phi \\ f_{\phi} &= -\frac{V U_{\phi} - U V_{\phi}}{V^2} \end{aligned}$$

5. Calculate the function g and its derivatives.

$$\begin{aligned} U &= f \cos \phi - g \sin \phi \\ V &= \frac{1}{2}(a - b) \sin 2\phi - h \cos 2\phi \\ g &= \tan \theta - U/V \end{aligned}$$

$$g_{\theta} = \sec^2 \theta$$

$$\begin{aligned} U_{\phi} &= -f \sin \phi - g \cos \phi \\ V_{\phi} &= (a - b) \cos 2\phi + 2h \sin 2\phi \\ g_{\phi} &= -\frac{V U_{\phi} - U V_{\phi}}{V^2} \end{aligned}$$

Your guesses of course, were wrong. You now have to find your errors $\Delta\theta$ and $\Delta\phi$ by solving $f_{\theta}\Delta\theta + f_{\phi}\Delta\phi = 0$ and $g_{\theta}\Delta\theta + g_{\phi}\Delta\phi = 0$. Thus the algorithm proceeds:

6. Calculate $\Delta\theta$ and $\Delta\phi$:

$$\text{Bottom} = \mathbf{f}_\theta \mathbf{g}_\phi - \mathbf{f}_\phi \mathbf{g}_\theta$$

$$\Delta\theta = (\mathbf{g}_\phi \mathbf{f} - \mathbf{f}_\phi \mathbf{g}) / \text{Bottom}$$

$$\Delta\phi = (\mathbf{f}_\theta \mathbf{g} - \mathbf{g}_\theta \mathbf{f}) / \text{Bottom}$$

7. Subtract these from your guesses:

$$\theta_{\text{new}} = \theta_{\text{old}} - \Delta\theta$$

$$\phi_{\text{new}} = \phi_{\text{old}} - \Delta\phi$$

8. Calculate $\frac{|\theta_{\text{new}} - \theta_{\text{old}}|}{\theta_{\text{old}}}$ and $\frac{|\phi_{\text{new}} - \phi_{\text{old}}|}{\phi_{\text{old}}}$

9. EITHER: Replace the old θ and ϕ with the new, and return to step 3.

OR, if $\frac{|\theta_{\text{new}} - \theta_{\text{old}}|}{\theta_{\text{old}}}$ and $\frac{|\phi_{\text{new}} - \phi_{\text{old}}|}{\phi_{\text{old}}}$ are as small as you wish them to be, STOP.

We already have an initial guess of $\theta = 48^\circ.1$, $\phi = 68^\circ.6$. I obtained

$$\theta = \underline{48^\circ.091694731928} \quad \phi = \underline{68^\circ.552263672895}$$

after 3 iterations. Again, the calculation was completed apparently instantaneously.