CHAPTER 2
Planes and Straight Lines

2.1 *Introduction*

In two dimensions, the simplest figure is a straight line. These is so little to be said about the geometry of a straight line that, when I dealt with the subject in orca.phys.uvic.ca/~tatum/celmechs/celms2.pdf , I was able to describe the geometry of the straight line in a mere eight pages and only 35 equations. Likewise, in three dimensions the simplest surface is a plane, and I don't suppose I'll need more than about 28 pages and 79 equations in this Chapter to describe it.

This chapter will involve a lot of numerical calculation. Before going further, I repeat the exhortation that I made in Preamble to these notes. This will make all the difference between very heavy and tedious work and instant gratification.

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2.2 *The Equation to a Plane*

The equation

$$Ax + By + Cz = 0 \qquad 2.2.1$$

represents a plane that contains the origin of coordinates.

The equation

$$Ax + By + Cz = D \qquad 2.2.2$$

represents a plane that does *not* contain the origin of coordinates. The four constants are not independent. If convenient in any particular situation, we could divide throughout by, for example, D (provided $D \neq 0$), to re-write the equation in the form

$$ax + by + cz = 1. \qquad 2.2.3$$

The equation can also be written the form

$$\frac{x}{x_0} + \frac{y}{y_0} + \frac{z}{z_0} = 1 \qquad 2.2.4$$

where $x_0 = 1/a,\quad y_0 = 1/b,\quad z_0 = 1/c$. The distances $x_0,\ y_0,\ z_0$ are then the distances where the three coordinate axes intersect the plane.

For those who like to keep track of units and dimensions (which should be everyone), we can suppose that x, y, z are distances, expressed in metres. The coefficients $ABCabc$ are each of dimension L^{-1}, expressible in m^{-1}, x_0, y_0, z_0 are of course lengths, of dimension L, and the constant D is dimensionless.

Three noncollinear points are sufficient to define a plane. For example, the equation to the plane that contains the points (2, 4, 7), (3, −5, 4), (6, 2, 3) is found by solving

$$\begin{aligned} 2a + 4b + 7c &= 1 \\ 3a - 5b + 4c &= 1 \\ 6a + 2b + 3c &= 1 \end{aligned} \qquad 2.2.5$$

This gives us, for the equation to the plane,

$$0.11278x - 0.03075y + 0.12782z = 1. \qquad 2.2.6$$

The equation to a plane containing the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) can be written as

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0, \qquad 2.2.7$$

although I am not sure that this is a faster way of computing the coefficients than the method used in the example above.

The orientation of a plane in space is best given by giving the direction cosines of a *normal* to the plane. In figure II.1, P is a point (x, y, z) in the plane $ax + by + cz = 1$. OM is a line from the origin, normal to the plane. Let p be the length of this normal, and let $(l,\ m,\ n)$ be its direction cosines. The coordinates of M are then $(pl,\ pm,\ pn)$

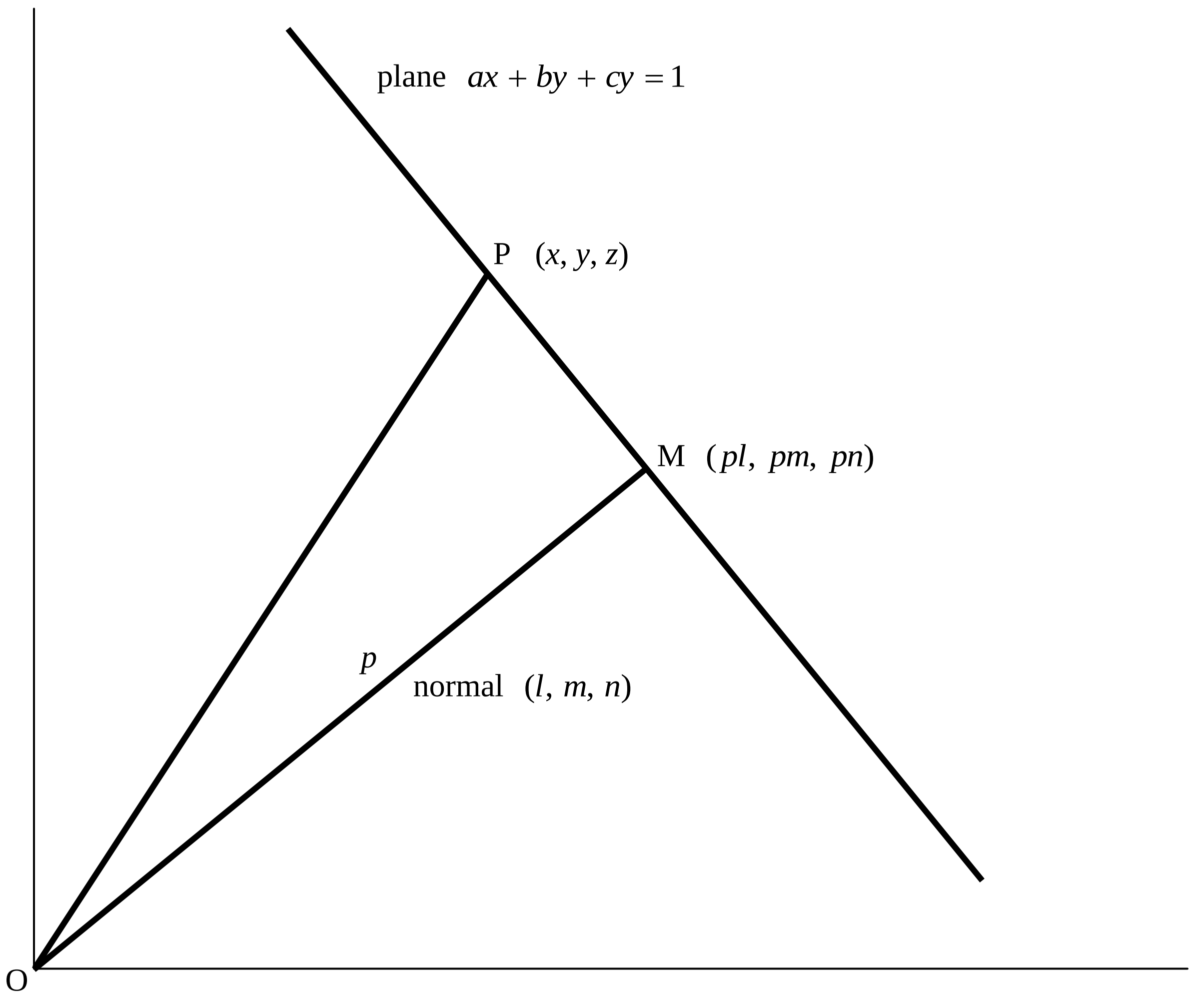


FIGURE II.1

The direction cosines of OM are $(l,\ m,\ n)$ and direction ratios of PM are $(x - pl,\ y - pm,\ z - pn)$. These lines are at right angles to each other, and therefore the scalar product of their direction ratios is zero: $l(x - pl) + m(y - pm) + n(z - pn) = 0.$ That is, $lx + my + nz = p(l^2 + m^2 + n^2) = p$. But since the point (x, y, z) already satisfies $ax + by + cz = 1$, we see that (a, b, c) are direction ratios of the normal to the plane $ax + by + cz = 1$, and the direction cosines of the normal to it are

$$\left(\frac{a}{\sqrt{a^2 + b^2 + c^2}},\ \frac{b}{\sqrt{a^2 + b^2 + c^2}},\ \frac{c}{\sqrt{a^2 + b^2 + c^2}} \right). \qquad 2.2.8$$

2.3 *Distance from the Origin to a Plane.*

Since the plane $ax + by + cz = 1$ does not contain the origin, it is of interest to know the perpendicular distance p from the origin to the plane. The coordinates of the point where the normal from the origin hits the plane (i.e. point M in the figure) are $(pl,\ pm,\ pn)$, and since M is in the plane it must satisfy $apl + bpm + cpn = 1$, from which we see that

$$p = \frac{1}{al + bm + cn}, \tag{2.3.1}$$

and hence, if we substitute the expressions in equation 2.2.8 for the direction cosines, we arrive at

$$p = \frac{1}{\sqrt{a^2 + b^2 + c^2}}. \tag{2.3.2}$$

It will be noted that this is also the perpendicular distance from the origin to the plane $ax + by + cz + 1 = 0$. The planes $ax + by + cz = \pm 1$ are parallel planes, at equal distances form the origin.

The coordinates of M are

$$\left(\frac{a}{a^2 + b^2 + c^2}, \quad \frac{b}{a^2 + b^2 + c^2}, \quad \frac{c}{a^2 + b^2 + c^2}\right). \tag{2.3.3}$$

(As always, check the dimensions of all these expressions.)

Example What is the perpendicular distance of the origin from the plane

$3x - 7y + 2z + 5 = 0$? (Assume all distances are expressed in m.) At what point does this perpendicular hit the plane? Where does the plane intersect the coordinate axes?

Answers. If we write the equation in the form $-\frac{3}{5}x + \frac{7}{5}y - \frac{2}{5}z = 1$ we find that

$$\underline{\underline{p = \frac{5}{\sqrt{62}} = 0.635 \text{ m}}} \tag{2.3.4}$$

and the coordinates of M are

$$\underline{\underline{x = -\tfrac{15}{62} = -0.242, \quad y = +\tfrac{35}{62} = 0.565, \quad z = -\tfrac{5}{31} = -0.161}} \quad \text{m} \tag{2.3.5}$$

The points where the plane intersects the coordinate axes are

$$\underline{\underline{(-\tfrac{5}{3},\ 0,\ 0),\quad (0,\ +\tfrac{5}{7},\ 0),\quad (0,\ 0,\ -\tfrac{5}{2})}} \qquad 2.3.6$$

2.4 *Distance from an Arbitrary Point to a Plane.*

What is the perpendicular distance of the point $(x_1,\ y_1,\ z_1)$ from the plane $ax + by + cz = 1$, and what are the coordinates $(x_2,\ y_2,\ z_2)$ of the point where the normal from $(x_1,\ y_1,\ z_1)$ hits the plane?

Direction ratios of the line joining the points $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$ are just $(x_1 - x_2,\ y_1 - y_2,\ z_1 - z_2)$, and direction ratios of the normal to the plane are $(a,\ b,\ c)$. These represent the same direction, and therefore are proportional to each other. That is

$$\begin{aligned} x_1 - x_2 &= ka \\ y_1 - y_2 &= kb \\ z_1 - z_2 &= kc \end{aligned} \qquad 2.4.1$$

The dimensions of k are L^2.

These, together with the observation that the point $(x_2,\ y_2,\ z_2)$ is in the plane, namely that

$$ax_2 + by_2 + cz_2 = 1, \qquad 2.4.2$$

show that

$$k = \frac{ax_1 + by_1 + cz_1 - 1}{a^2 + b^2 + c^2} \qquad 2.4.3$$

and hence, after a little algebra,

$$\begin{aligned} x_2 &= \frac{x_1(b^2 + c^2) - a(by_1 + cz_1 - 1)}{a^2 + b^2 + c^2} \\ y_2 &= \frac{y_1(c^2 + a^2) - b(cz_1 + ax_1 - 1)}{a^2 + b^2 + c^2} \\ z_2 &= \frac{z_1(a^2 + b^2) - c(ax_1 + by_1 - 1)}{a^2 + b^2 + c^2} \end{aligned} \qquad 2.4.4$$

The perpendicular distance p from the point to the plane, which is just the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is rather easier, for it is just given by

$$p^2 = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2] = k^2(a^2 + b^2 + c^2)\,; \qquad 2.4.5$$

that is:

$$p = \frac{ax_1 + by_1 + cz_1 - 1}{\sqrt{a^2 + b^2 + c^2}}\,. \qquad 2.4.6$$

Example How far from the plane $3x - 7y + 2z + 5 = 0$ are each of the following four points?

O: (0, 0, 0)
A: (5, 2, 1)
B: (4, 3, 2)
C: (7, 5, 3)

Note that, if we write the equation to the plane in the form $ax + by + cz = 1$, our particular plane becomes $-\frac{3}{5}x + \frac{7}{5}y - \frac{2}{5}z = 1$.

Answers. By application of equation 2.4.6 for p, and taking the symbol $\sqrt{\ }$ to mean the positive square root, we find the following distances of these four points from the plane:

O: $p = -0.635$ m
A: $p = -1.016$ m
B: $p = 0.000$ m
C: $p = +0.381$ m

We see that the distance of B from the plane is zero. That means, of course, that B is contained in the plane. But what do the signs on the others mean? Not very much for a single point, and we can quite happily and correctly say that the distance of the point A from the plane is 0.016 m. However, we can interpret the signs as follows: The distance of A from the plane is negative; B is in the plane, and its distance from the plane is zero; the distance of C from the plane is positive. This means that A and C are on opposite sides of the plane. Likewise, A is on the same side of the plane as the origin, while C is on the opposite side of the plane from the origin.

Thus is general, if we have a plane $ax + by + cz = 1$ and two points (x_1, y_1, z_1), (x_2, y_2, z_2), these two points are on the same side of the plane if $ax_1 + by_1 + cz_1 - 1$ and $ax_2 + by_2 + cz_2 - 1$ have the same sign, and they are on opposite sides of the plane if these two expressions are of opposite sign.

We take the opportunity here of calculating the coordinates of the points M where the perpendiculars from these four points hit the plane:

$$
\begin{array}{llll}
\mathrm{M_O}: & (-0.242, & +0.565, & +0.161) \\
\mathrm{M_A}: & (+4.613, & +2.903, & +0.742) \\
\mathrm{M_B}: & (+4.000, & +3.000, & +2.000) \\
\mathrm{M_C}: & (+7.145, & +4.661, & +3.097)
\end{array}
$$

2.5 *Two parallel planes*

Here are two parallel planes:

$$
\begin{aligned}
\mathrm{P_1}: \quad & 3x - 7y + 2z + 5 = 0 \\
\mathrm{P_2}: \quad & 3x - 7y + 2z + 9 = 0
\end{aligned}
\qquad 2.5.1
$$

Two questions:

How far apart are they?
Do any of the following four points lie between these planes?

$$
\begin{array}{ll}
\mathrm{O}: & (0,\ 0,\ 0) \\
\mathrm{A}: & (-4,\ -2,\ +2) \\
\mathrm{B}: & (-5,\ -3,\ -6) \\
\mathrm{C}: & (-3,\ -2,\ -8)
\end{array}
$$

The point $(-3,\ 0,\ 0)$ is in the plane $\mathrm{P_2}$. Direction ratios to the plane are $(3,\ -7,\ 2)$. Therefore the normal to the plane, passing through the point $(-3,\ 0,\ 0)$, is given by the equations

$$\frac{x+3}{3} = \frac{y}{7} = \frac{z}{2} \qquad 2.5.2$$

This line hits plane $\mathrm{P_1}$ at a point given by the solution of these two equations together with

$$3x - 7y + 2z + 5 = 0. \qquad 2.5.3$$

That is, at the point $(-\frac{30}{9},\ -\frac{7}{9},\ -\frac{2}{9})$

The distance between this point and $(-3,\ 0,\ 0)$ is the distance between the planes. That is, $\underline{\underline{6.385\ \mathrm{m}}}$.

To answer the second question, let's first of all calculate the distances of each of the four points from the two planes. Here they are, although to answer our question, we only need look at the signs.

	P_1	P_2
O	-0.635	-1.143
A	-1.397	-1.905
B	$+0.127$	-0.391
C	$+0.762$	$+0.254$

We see from this that the point B is between the two planes.

2.6 *Two Planes, Not Necessarily Parallel*

The left hand side of the following equation 2.6.1 can be written (not necessarily easily) as the product of two linear expressions, and thus equation 2.6.1 represents two planes.

$$12x^2 - 20y^2 - 21z^2 + 47yz - 36zx + 14xy + 14x - 23y - 5z + 6 = 0. \qquad 2.6.1$$

How do we find the two linear factors? This is how I did it - maybe there's an easier way. Let me know if you think of one.

By successively putting $y = z = 0, \quad z = x = 0, \quad x = y = 0$, I found that the planes contain the following six points on the coordinate axes:

$$\begin{array}{ll} A_1: \ (+\tfrac{1}{3}, 0, 0) & A_2: \ (+\tfrac{3}{2}, 0, 0) \\ B_1: \ (0, +\tfrac{3}{4}, 0) & B_2: \ (0, -\tfrac{2}{5}, 0) \\ C_1: \ (0, 0, +\tfrac{2}{5}) & C_2: \ (0, 0, -\tfrac{3}{7}) \end{array}$$

There are four ways in which two planes can contain these six points. I list them below, together with the equations that represent them:

$$\begin{array}{ll} A_1B_1C_1: 18x + 8y + 9z - 6 = 0 & A_2B_2C_2: \ 4x - 15y - 14z - 6 = 0 \\ A_1B_2C_1: \ 6x - 5y + 3z - 2 = 0 & A_2B_1C_2: \ 2x + 4y - 7z - 3 = 0 \\ A_1B_1C_2: \ 9x + 4y - 7z - 3 = 0 & A_2B_2C_1: \ 4x - 15y + 9z - 6 = 0 \\ A_1B_2C_2: 18x - 15y - 14z + 6 = 0 & A_2B_1C_1: \ 4x + 8y + 9z - 6 = 0 \end{array} \qquad 2.6.2$$

Only the second of these pairs of linear equations, when multiplied together, yield the original quadratic equation, and these are therefore the sought-after factors. The original quadratic equation 2.6.1 therefore represents the two planes

$$\begin{aligned} 6x - 5y + 3z - 2 &= 0 \\ 2x + 4y - 7z - 3 &= 0 \end{aligned} \qquad 2.6.3$$

Let us now try four more examples, each of which has a slightly different wrinkle.

Example 2.

$$9x^2 + 49y^2 + 4z^2 - 28yz + 12zx - 42xy + 42x - 98y + 28z + 45 = 0. \qquad 2.6.4$$

By successively putting $y = z = 0$, $z = x = 0$, $x = y = 0$, it is found that the planes contain the following six points on the coordinate axes:

$$\begin{array}{llll} A_1: & (-3, 0, 0) & A_2: & (-\frac{5}{3}, 0, 0) \\ B_1: & (0, \frac{5}{2}, 0) & B_2: & (0, \frac{9}{7}, 0) \\ C_1: & (0, 0, -\frac{9}{2}) & C_2: & (0, 0, -\frac{5}{2}) \end{array}$$

Again, there are four possible pairs of planes that connect six points. I list only the pair $A_1B_2C_1$ and $A_2B_1C_2$:

$$A_1B_2C_1: \; 3x - 7y + 2z + 9 = 0 \qquad A_2B_1C_2: \; 3x - 7y + 2z + 5 = 0. \qquad 2.6.5$$

These are the two parallel planes that we dealt with in Section 2.5, and which we found to be separated by 6.385 m. The other three pairs of planes that can connect the six points yield linear expressions which, when multiplied, do not yield the original quadratic expression.

Example 3.

Next, let us try

$$4x^2 + 9y^2 + 25z^2 - 30yz - 20zx + 12xy + 24x + 36y - 60z + 36 = 0. \qquad 2.6.6$$

This turns out to be

$$(2x + 3y - 5z + 6)^2 = 0$$

which you could describe as two coincident planes, or, if you prefer, just one plane.

The reader who has worked through these examples in detail will have discovered that it is a good deal more laborious than might be apparent to a more casual reader. The writer (jtatum at uvic dot ca) would be grateful if anyone can suggest a quicker way of doing the factorization.

Lastly, two more:

Example 4.

$$12x^2 - 2y^2 - 24z^2 + 26yz + 68zx - 2xy + 6x + 2y - 3z + 10 = 0 \qquad 2.6.7$$

and

Example 5

$$3x^2 + y^2 - 4z^2 + yz - 5zx + 9xy - 2x - 6y + 5z + 4 = 0. \qquad 2.6.8$$

Don't spend time on these – you will not be able to split them into two real linear factors. Neither of them represents two planes. For one thing, neither of them crosses the x-axis in any real points. (Try it and see.) In fact the first equation represents a paraboloid, and the second represents an ellipsoid. This raises the question, given an equation of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0, \qquad 2.6.9$$

how do I know whether it can be factored into two real linear factors, representing two planes? We shall discuss this in a later chapter. In the meantime let us note (without yet proof) that a necessary (but not sufficient) condition that it can be factored into two real linear factors is that

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0. \qquad 2.6.10$$

You might like to evaluate this determinant (which we shall later denote as Δ_3) for each of the above five examples. The first three are zero, as they must be for two planes. The fourth is also zero, and, although this is a necessary condition for the equation to represent two planes, it is not a sufficient condition. The last of the five is nonzero.

Let us now return to the two planes

$$\begin{aligned} 6x - 5y + 3z - 2 &= 0 \\ 2x + 4y - 7z - 3 &= 0 \end{aligned} \qquad \text{was 2.6.3, now is} \quad 2.6.11$$

They are not parallel. I have two questions: **1.** What is the angle between the two planes? **2.** The two planes intersect in a line. What are the direction cosines of this line?

Answers:

1. The direction cosines of the normals to the two planes can be thought of as two vectors, **A** and **B:**

A = (6, –5, 3) and **B** = (2, 4, –7)

and the angle θ between them can be found by the usual definition of the dot product:

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$$

Thus $$\underline{\theta = 114^\circ.6628}$$

if you go clockwise from **A** to **B.** Otherwise, it's 65º3372 .

2.

By successively putting $x = 0,\ y = 0,\ z = 0,$ we can see that the line represented by the two equations intersects the $yz-,\ zx-,\ xy-$planes at $(0,\ -1,\ -1),\ \left(\frac{23}{48},\ 0,\ -\frac{7}{24}\right),\ \left(\frac{23}{34},\ \frac{7}{17},\ 0\right)$ respectively. Direction ratios of the line can be found from any pair of these three points. [It will be recalled that if we have two points (a_1, b_1, c_1), (a_2, b_2, c_2), then the line joining them has direction ratios $(a_1 - a_2,\ b_1 - b_2,\ c_1 - c_2)$.] In integers, direction ratios of the lines are $(23,\ 48,\ 34)$, and the direction cosines are therefore $\underline{\underline{(0.36416,\ 0.75999,\ 0.53833)}}$.

Consider now the two planes

$$\begin{aligned} A_1x + B_1y + C_1z + D_1 &= 0 \\ A_2x + B_2y + C_2z + D_2 &= 0 \end{aligned} \qquad 2.6.12$$

By successively putting $x = 0,\ y = 0$, we can see that the line intersects the *yz*-plane at the point

$$\left(x_1 = 0,\ y_1 = \frac{C_1D_2 - C_2D_1}{B_1C_2 - B_2C_1},\ z_1 = \frac{-D_1 - B_1y_1}{C_1}\right), \qquad 2.6.13$$

and it intersects the *zx*-plane at the point

$$\left(x_2 = \frac{C_1D_2 - C_2D_1}{A_1C_2 - A_2C_1},\ y_2 = 0,\ z_2 = \frac{-D_1 - A_1x_2}{C_1} \right). \qquad 2.6.14$$

Direction ratios of the line connecting these two points [i.e. of the line of intersection of the two planes (provided that they are not parallel)] are

$$(x_2,\ -y_1,\ z_2 - z_1) \qquad 2.6.15$$

and the direction cosines are

$$\left(\frac{x_2}{\sqrt{x_2^2 + y_1^2 + (z_2 - z_1)^2}},\ \frac{-y_1}{\sqrt{x_2^2 + y_1^2 + (z_2 - z_1)^2}},\ \frac{z_2 - z_1}{\sqrt{x_2^2 + y_1^2 + (z_2 - z_1)^2}} \right). \qquad 2.6.16$$

The next section will be very much easier to follow if you were to program your computer to calculate these direction cosines instantly for any two planes.

2.7 *Three Planes*

Unless two or all three of three planes are parallel, in general three planes will intersect at a single point. Consider, for example, the three planes

$$\begin{aligned} 2x + 3y + 4z - 9 &= 0 \\ x + y - 8z + 6 &= 0 \\ 5x + 6y - 12z + 1 &= 0 \end{aligned} \qquad 2.7.1$$

They obviously intersect at the point (1, 1, 1).

But now consider the following three planes, which I refer to with the letters **a**, **b**, **c**:

$$\begin{aligned} \mathbf{a}:\ 2x + 3y + 4z - 9 &= 0 \\ \mathbf{b}:\ x + y - 8z + 6 &= 0 \\ \mathbf{c}:\ 5x + 6y - 20z + 12 &= 0 \end{aligned} \qquad 2.7.2$$

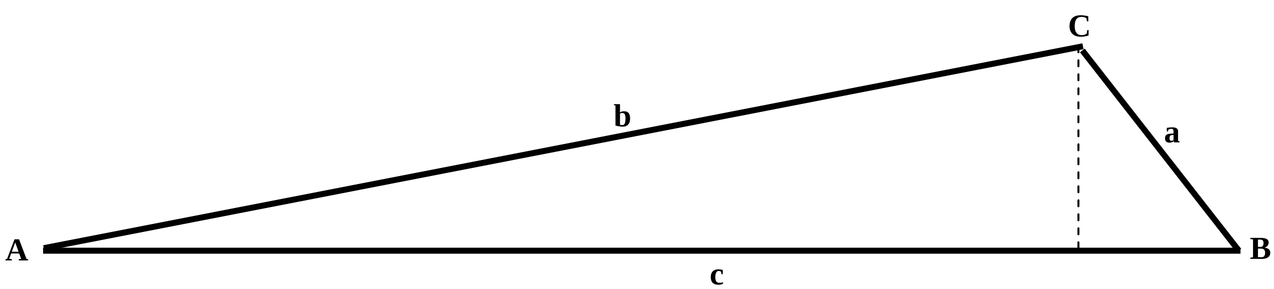
You may have trouble solving these. The problem is that the determinant of the coefficients is zero, which means that any one of the equations is a linear combination of the other two. For example, $\mathbf{a} + 3\mathbf{b} = \mathbf{c}$.

If you calculate the direction cosines of the line of intersection of any two of these planes (see Section 2.6), you will find that the <u>direction cosines of the line joining any</u>

two pairs are the same, namely $(+0.813390051,\ -0.580992894,\ +0.029049645)$. That means that the three planes enclose a triangular prism.

For convenience I refer to the line of intersection of planes **b** and **c** with the letter **A**
and the line of intersection of planes **c** and **a** with the letter **B**
and the line of intersection of planes **a** and **b** with the letter **C**
These lines are the edges of the triangular prism.

The angle between the planes **b** and **c** is 11º.28.
The angle between the planes **c** and **a** is 51º.89.
The angle between the planes **a** and **b** is 116º.73.
Thus the shape of the triangular cross-section of the prism is like this:



I haven't yet given any indication of its size. To find the size, let us consider any point on the line (*sic*) **C**. For example the point $(1,\ 1,\ 1)$ is on the line **C**. From Section 2.4, we know how to calculate its distance from the plane **c** – i.e. the dashed line in the figure. We find that the distance of **C** from **c** is 0.1397. Of course, we obtain the same result if we start with any point on the line **C** such as the points $(-27,\ +21,\ 0)$ or $(-0.4,\ +2,\ +0.95)$, or any other point that lies on **C.**

Having found the distance of **C** from **c**, we can immediately find the sides of the triangle by elementary means. Thus we find that

The distance between the edges **B** and **C** is 0.1776.
The distance between the edges **C** and **A** is 0.7080.
The distance between the edges **A** and **B** is 0.8037.

The calculation can (and should) be checked by finding the distances of **A** from **a**, and of **B** from **b.** These should result in the same length of the sides of the triangle.

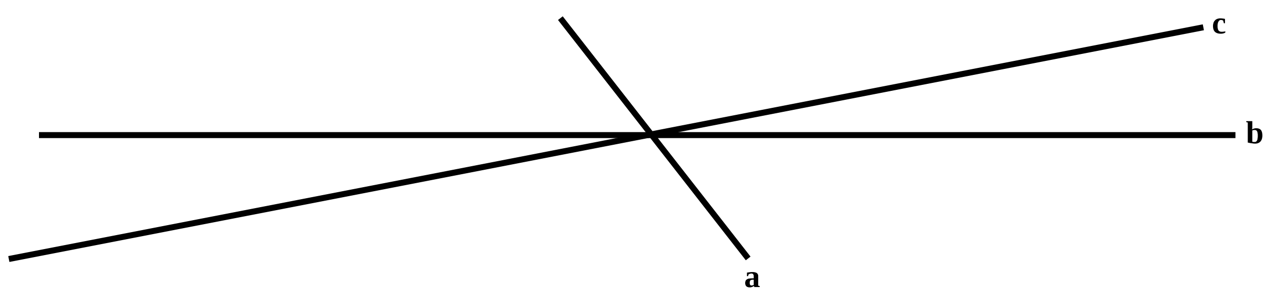
Now consider the three planes

$$\begin{aligned}
\mathbf{a}:&\ \ 2x + 3y + 4z - 9 = 0 \\
\mathbf{b}:&\ \ \ \ x + y - 8z + 6 = 0 \\
\mathbf{c}:&\ \ 5x + 6y - 20z + 9 = 0
\end{aligned} \qquad 2.7.3$$

These differ from the previous three only by the constant term in the third equation. In other words, all we have done is to translate the plane **c** without rotation. Obviously, then, we find the angles between the three planes are as before:

The angle between the planes **b** and **c** is 11º.28.
The angle between the planes **c** and **a** is 51º.89.
The angle between the planes **a** and **b** is 116º.73.

Now let us find, by the same method as above, the distance between **C** and **c.** We find that this distance is zero. Likewise we shall find that the distances between **A** and **a** and between **B** and **b** are zero. The three planes are like this:



In this case the three planes intersect in a single line.

Summary of the last three examples

The three planes
$$\begin{aligned}
2x + 3y + 4z - 9 &= 0 \\
x + y - 8z + 6 &= 0 \\
5x + 6y - 12z + 1 &= 0
\end{aligned} \qquad 2.7.1$$

intersect at a single point.

The three planes
$$\begin{aligned} 2x + 3y + 4z - 9 &= 0 \\ x + y - 8z + 6 &= 0 \\ 5x + 6y - 20z + 12 &= 0 \end{aligned} \qquad 2.7.2$$

enclose a triangular prism.

The three planes
$$\begin{aligned} 2x + 3y + 4z - 9 &= 0 \\ x + y - 8z + 6 &= 0 \\ 5x + 6y - 20z + 9 &= 0 \end{aligned} \qquad 2.7.3$$

enclose a prism of zero area; i.e. they intersect in a straight line.

Suppose that you work in algebra rather than with particular numbers, what about the following three planes, in which we assume that no two of the planes are parallel?

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_2x + b_2y + c_2z + d_2 &= 0 \\ a_3x + b_3y + c_3z + d_3 &= 0 \end{aligned} \qquad 2.7.4$$

Define:
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{and} \quad \Delta' = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}. \qquad 2.7.5$$

you should find, if you work through the algebra in the same way as we did the numerical examples, that:

1. If $\Delta \neq 0$, the three planes intersect at a single point.

2. If $\Delta = 0$ and $\Delta' \neq 0$, the three planes enclose a triangular prism.

3. If $\Delta = 0$ and $\Delta' = 0$, the three planes intersect in a common line.

2.8 *Straight Lines*

A straight line is represented by the intersection of two (nonparallel) planes.

We saw in Section 2.6 that the following two equations (each of which represents a plane:

$$\begin{aligned} A_1x + B_1y + C_1z + D_1 &= 0 \\ A_2x + B_2y + C_2z + D_2 &= 0 \end{aligned} \qquad 2.8.1$$

together represent a straight line. We also determined in Section 2.6 that direction ratios of this line are:

$$\left(\frac{C_1D_2 - C_2D_1}{A_1C_2 - A_2C_1}, \quad \frac{C_1D_2 - C_2D_1}{B_2C_1 - B_1C_2}, \quad \left(\frac{C_1D_2 - C_2D_1}{C_1}\right)\left(\frac{B_1}{B_1C_2 - B_2C_1} - \frac{A_1}{A_1C_2 - A_2C_1}\right) \right) \qquad 2.8.2$$

This can be written, for ease of computation, as

$$\left(\frac{p}{q}, \quad \frac{p}{r}, \quad -\frac{p}{C_1}\left(\frac{B_1}{r} + \frac{A_1}{q}\right)\right), \qquad 2.8.3$$

where $$p = C_1D_2 - C_2D_1, \qquad 2.8.4$$

$$q = A_1C_2 - A_2C_1, \qquad 2.8.5$$

$$r = B_2C_1 - B_1C_2 \quad . \qquad 2.8.6$$

Thus, for example, direction ratios of the line of intersection of the two planes

line 1: $$\begin{aligned} 6x - 5y + 3z - 2 &= 0 \\ 2x + 4y - 7z - 3 &= 0 \end{aligned} \qquad 2.8.7$$

are $$(0.4791\dot{6}, \quad 1, \quad 0.708\dot{3}),$$

and the direction cosines are $(0.36416, \quad 0.75999, \quad 0.53833)$,
in agreement with what we obtained in Section 2.6.

[By the way, you may have noticed that I use the expression "direction ratios" but "the direction cosines". This is not accidental.]

Now let us consider another line, namely the line

line 2:
$$\begin{aligned} 4x + 3y - 2z + 7 &= 0 \\ 3x + 2y + 5z - 2 &= 0 \end{aligned} . \qquad 2.8.8$$

Its direction cosines are

$$(-0.58973, \quad 0.80700, \quad 0.03104).$$

The next question that might occur to us is: Where do the two lines 2.8.7 and 2.8.8 intersect? This is immediately followed by the question: *Do* they intersect? In three dimensions two lines in general will *not* intersect. Two lines will intersect at a point iff:

1. They are coplanar.
2. They are not parallel.

Otherwise the two lines are said to be *skew*, in which case we shall want to know: What is the least distance between the two lines? If you can think vividly in three dimensions, you will probably agree that the shortest distance between two skew lines is along a third line joining the two and perpendicular to each. The calculation is fairly long, though trivial by computer.

To help to visualize the two lines, let us note that **line 1**intersects the coordinate planes at

$$\begin{array}{lll} (0, & -1, & 1) \\ (0.4791\dot{6}, & 0, & -0.291\dot{6}) \\ (0.67647059, & 0.41176471, & 0 \quad) \end{array}$$

or, for short:

$$\begin{array}{lll} (0.00, & -1.00, & 1.00) \\ (0.48, & 0.00, & -0.29) \\ (0.68, & 0.41, & 0.00) \end{array} .$$

And **line 2** intersects the coordinate planes at

$$\begin{array}{lll} (\ 0, & -1.63157895, & 1.05263158) \\ (-1.19230769, & 0, & 1.11548462) \\ (\ 5, & -6.5, & 0 \quad) \end{array}$$

or, for short:

$$\begin{array}{lll}(\;\;0.00, & -1.63, & 1.05) \\ (-1.19, & 0.00, & 1.12) \\ (\;\;5.00, & -6.50, & 0.00)\end{array} .$$

It may be more convenient to write the equations to the two lines in the forms

line 1: $$\frac{x-a_1}{l_1} = \frac{y-b_1}{m_1} = \frac{z-c_1}{n_1} \qquad 2.8.9$$

line 2: $$\frac{x-a_2}{l_2} = \frac{y-b_2}{m_2} = \frac{z-c_2}{n_2} . \qquad 2.8.10$$

Here (a_1, b_1, c_1) is some point that we know to be on line 1
(a_2, b_2, c_2) is some point that we know to be on line 2
(l_1, m_1, n_1) are the direction cosines of line 1
(l_2, m_2, n_2) are the direction cosines of line 2

In our particular example, we may choose
$a_1 = 0.00, \;\; b_1 = -1.00, \;\; c_1 = 1.00$
$a_2 = 5.00, \;\; b_2 = -6.50, \;\; c_2 = 0.00$
and the direction cosines are
$l_1 = \;\; 0.36416, \;\; m_1 = 0.75999, \;\; n_1 = 0.53833$
$l_2 = -0.58973, \;\; m_2 = 0.80700, \;\; n_2 = 0.03104$

Let $\mathrm{P}_1(a_1 + \lambda_1 l_1, \;\; b_1 + \lambda_1 m_1, \;\; c_1 + \lambda_1 n_1)$ be some other point on line 1, and
let $\mathrm{P}_2(a_2 + \lambda_2 l_2, \;\; b_2 + \lambda_2 m_2, \;\; c_2 + \lambda_2 n_2)$ be some other point on line 2.

Direction ratios of $\mathrm{P}_1\mathrm{P}_2$ are

$$(a_1 - a_2 + \lambda_1 l_1 - \lambda_2 l_2, \;\; b_1 - b_2 + \lambda_1 m_1 - \lambda_2 m_2, \;\; c_1 - c_2 + \lambda_1 n_1 - \lambda_2 n_2) \qquad 2.8.11$$

If $\mathrm{P}_1\mathrm{P}_2$ is to be the shortest distance between line 1 and line 2, it must be perpendicular to each. That is, the dot product of the direction ratios of $\mathrm{P}_1\mathrm{P}_2$ with each line must be zero:

$$l_1(a_1 - a_2 + \lambda_1 l_1 - \lambda_2 l_2) + m_1(b_1 - b_2 + \lambda_1 m_1 - \lambda_2 m_2) + n_1(c_1 - c_2 + \lambda_1 n_1 - \lambda_2 n_2) = 0 \qquad 2.8.12$$

and

$$l_2(a_1 - a_2 + \lambda_1 l_1 - \lambda_2 l_2) + m_2(b_1 - b_2 + \lambda_1 m_1 - \lambda_2 m_2) + n_2(c_1 - c_2 + \lambda_1 n_1 - \lambda_2 n_2) = 0. \qquad 2.8.13$$

These can be written:

$$(l_1^2 + m_1^2 + n_1^2)\lambda_1 - (l_1 l_2 + m_1 m_2 + n_1 n_2)\lambda_2 + l_1(a_1 - a_2) + m_1(b_1 - b_2) + n_1(c_1 - c_2) = 0 \qquad 2.8.14$$

and

$$(l_1 l_2 + m_1 m_2 + n_1 n_2)\lambda_1 - (l_2^2 + m_2^2 + n_2^2)\lambda_2 + l_2(a_1 - a_2) + m_2(b_1 - b_2) + n_2(c_1 - c_2) = 0 \qquad 2.8.15$$

Provided that $l_1 m_1 n_1 l_2 m_2 n_2$ are the direction *cosines* and not merely direction *ratios*, these reduce to

$$\lambda_1 - (l_1 l_2 + m_1 m_2 + n_1 n_2)\lambda_2 + l_1(a_1 - a_2) + m_1(b_1 - b_2) + n_1(c_1 - c_2) = 0 \qquad 2.8.16$$

$$(l_1 l_2 + m_1 m_2 + n_1 n_2)\lambda_1 - \lambda_2 + l_2(a_1 - a_2) + m_2(b_1 - b_2) + n_2(c_1 - c_2) = 0 \qquad 2.8.17$$

To solve these for λ_1 and λ_2 by computer is trivial, although seriously tedious by hand.

In our example, these equations are

$$\lambda_1 - 0.41527\lambda_2 + 2.89747 = 0 \qquad 2.8.18$$

$$0.41527\lambda_1 - \lambda_2 + 7.41821 = 0, \qquad 2.8.19$$

with solutions

$$\lambda_1 = 0.22120, \quad \lambda_2 = 7.51007 \qquad 2.8.20$$

The coordinates of P_1 and P_2 are therefore

P_1: (0.08055, –0.83189, 1.11908)
P_2: (0.57107, –0.43936, 0.23310)

and the distance between them, which is the shortest distance between the two lines, is 1.08611. Of course, if the coordinates of P_1 and P_2 turned out to be identical, this would mean that the two lines intersect at that point.

2.9 *Finding a Meteorite*

We end this chapter with three astronomical examples, all to do with meteors or meteorites, the first in this section 2.9, and a second in the next section 2.10, and a third in Section 2.11.

We suppose that two observers are situated on the surface of the Earth and separated by a few tens of kilometres. Both observe a meteor in the sky, and each of them makes measurements of the altitude and azimuth of two points along the apparent path of the meteor on the sky. The object is to find the true path of the meteor through the atmosphere, and to predict where the meteorite will land.

We shall use a Flat Earth approximation. Towards the end of the section we shall briefly discuss how good or how bad such an approximation is.

We shall call the observers O and A. In what follows, all distances will be assumed to be expressed in km. Observer O is at the origin of a rectangular coordinate system, and the coordinates of observer A referred to this system are (a, b, 0), as shown in figure II.1. Observer A will refer his own observations to his local coordinate axes indicated in figure II.1 in red.

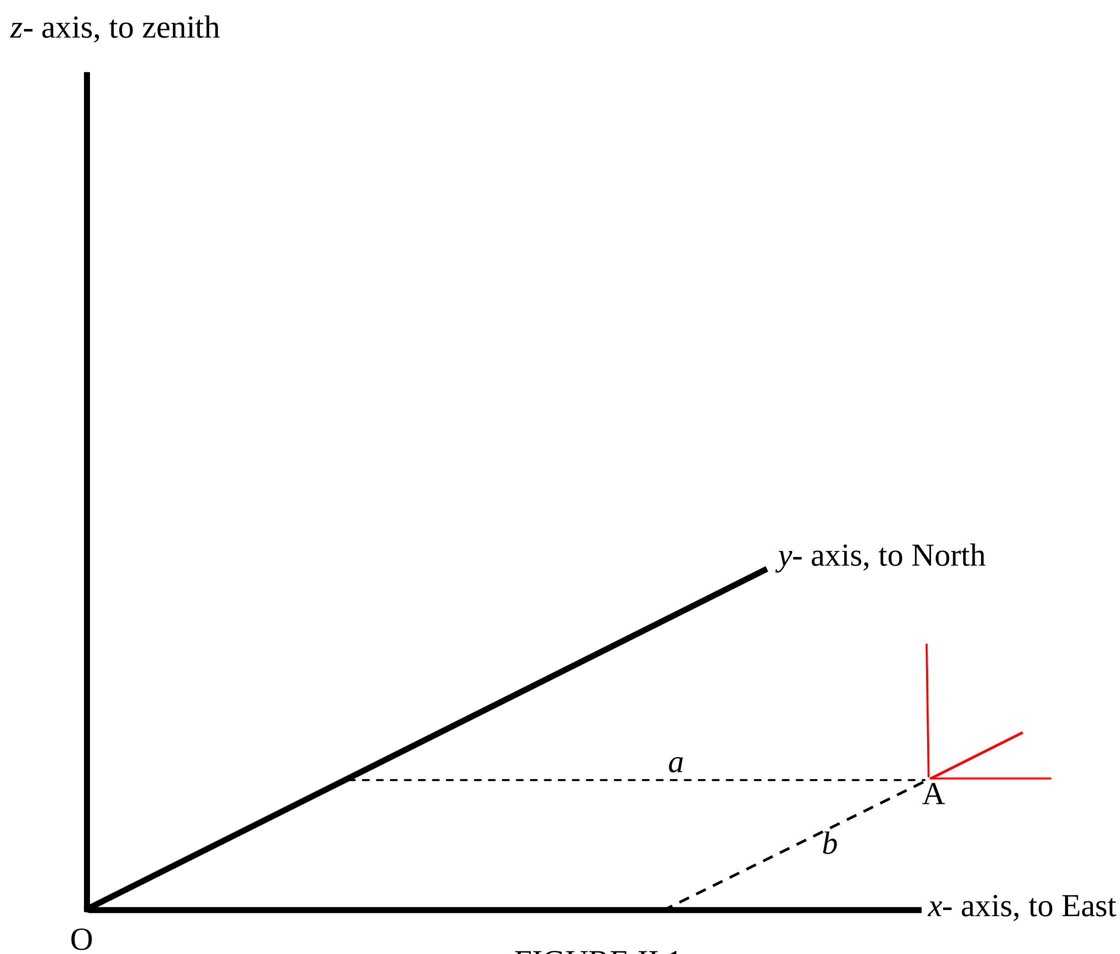


FIGURE II.1

In figure II.2, observer O measures the zenith distance θ and the azimuth ϕ of two points M and N on the apparent path of the meteor. Azimuths are expressed counterclockwise from the *x*- axis in the usual convention for spherical coordinates.

These measurements show that the path of the meteor is in the plane OM_1M_2. The rectangular coordinates of these three points are as follows:

O: $(0, 0, 0)$
M_1: $(r_1 \sin\theta_1 \cos\phi_1,\ \ r_1 \sin\theta_1 \sin\phi_1,\ \ r_1 \cos\theta_1)$
M_2: $(r_2 \sin\theta_2 \cos\phi_2,\ \ r_2 \sin\theta_2 \sin\phi_1,\ \ r_2 \cos\theta_2)$

The equation to the plane OM_1M_2 is immediately given by equation 2.2.7, repeated here as equation 2.8.1, in which the $x_1, y_1, \ldots$ etc are the coordinates given above for the three points.

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \qquad \text{was 2.2.7, is now} \qquad 2.9.1$$

This is an equation of the form

$$A_1 x + B_1 y + C_1 z + D_1 = 0 \qquad 2.9.2$$

in which the coefficients are functions of the measured angles, and which do not, it will be noted with relief, contain the unknown distances r_1 and r_2.

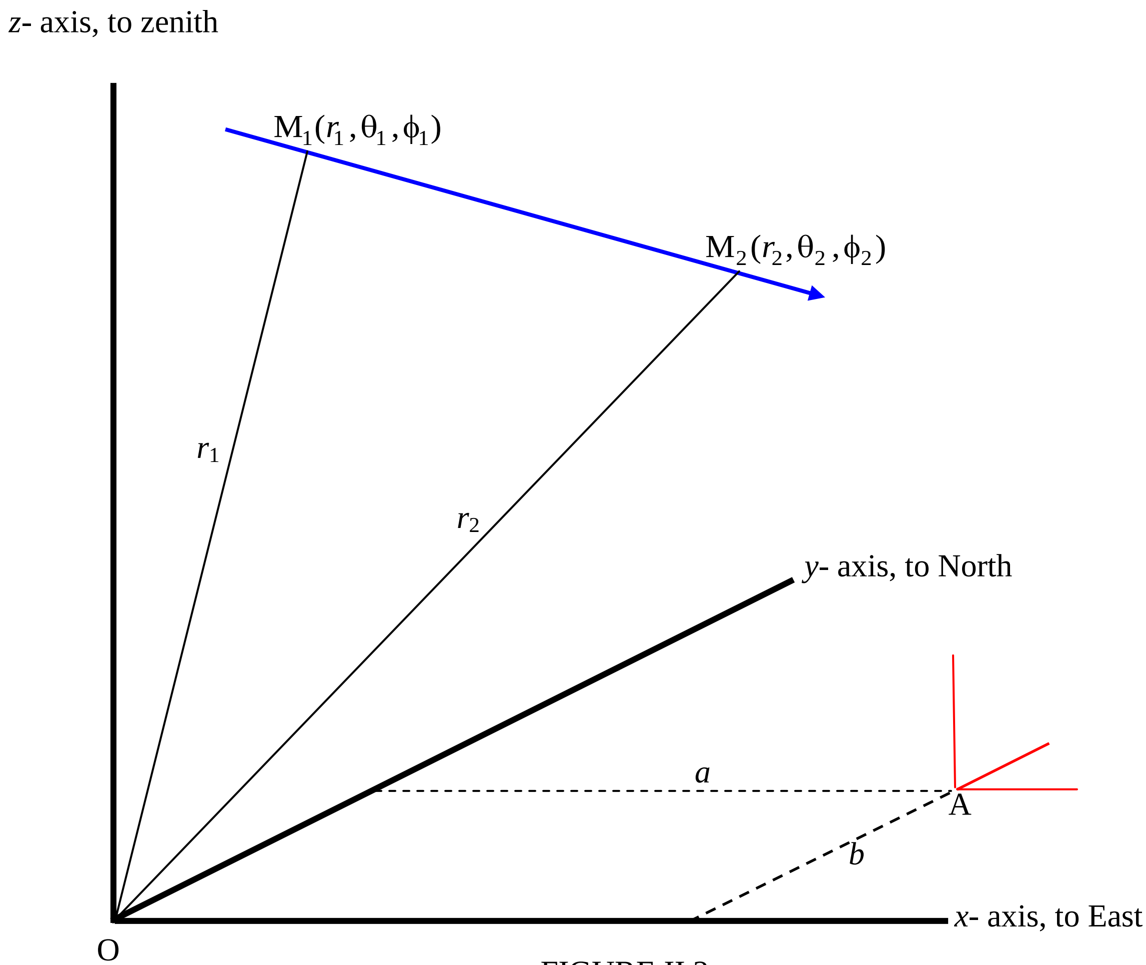


FIGURE II.2

At the same time, observer A also measures the zenith distance and azimuth of two points N_1 and N_2 on the meteor track, referred to his (red) coordinate system. (These need not be, and indeed probably will not be, the same two points that O measured.). We'll call the angles measured with respect to the red coordinate axes (θ'_1, ϕ'_1) for N_1 and (θ'_2, ϕ'_2) for N_2, and we'll denote the unknown distance of N_1 and N_2 from A by r'_1 and r'_2.

The meteor is now known to be in the plane AN_1N_2, and the coordinates of these three points referred to O as origin are

A: $(a, b, 0)$
N_1: $(a + r'_1 \sin\theta'_1 \cos\phi'_1,\ \ b + r'_1 \sin\theta'_1 \sin\phi'_1,\ \ r'_1 \cos\theta'_1)$
N_2: $(a + r'_2 \sin\theta'_2 \cos\phi'_2,\ \ b + r'_2 \sin\theta'_2 \sin\phi'_2,\ \ r'_2 \cos\theta'_2)$

By application of equation 2.9.1, we now know the equation to the plane AN_1N_2, referred to O as origin of coordinates. We'll call this equation

$$A_2x + B_2y + C_2z + D_2 = 0\,, \qquad 2.9.3$$

in which, as before, the coefficients contain only the measured angles and not the unknown distances. With equations 2.9.2 and 2.9.3, we now know the path of the meteor through the sky.

If you eliminate z from these two equations, the result will be an equation of the form

$$ax + by + c = 0. \qquad 2.9.4$$

This is the equation of the projection of the meteor's path on the ground. If any material has reached the ground, it will be along this ground track.

If you put $z = 0$ in equations 2.9.2 and 2.9.3, and solve the resulting equations for x and y, this will give the ground coordinates of where the meteorite hit the ground, assuming it continued to travel in a straight line. Presumably the meteorite will actually fall somewhere along the line 2.9.4, but short of this point.

Further details of this method, with a numerical example, are to be found in JRASC **92,**78, (1988).

The Flat Earth approximation assumes that the height of any observer above sea-level is small compared with the height of the meteor above sea level. The latter is typically of the order of 100 km. It also assumes that the distance the A is below the tangent plane at O is also small compared with the height of the meteor. If the distance between O and A is about 25, km, the distance of A below the tangent plane at O is about 0.1 km. Whether the Flat Earth approximation is good enough depends upon the precision of the measurement, and on how precise a solution is desired. For visual estimates of zenith distances and azimuths by surprised observers, the Flat Earth approximation is more than adequate. If two large-scale photographs are obtained and measured precisely, the sphericity of the Earth is just one of a number of other factors that have to be taken into account.

2.10 *The Widmanstätten Pattern.*

An octahedrite meteorite consists of alternating plane lamellae parallel to the faces of a regular octahedrite. When you make a plane slice through an octahedrite, you see, in the slice, a pattern of lines, known as the Widmanstätten pattern. These lines are the intersection of the octahedrite faces with the plane of your slice. We'll see in this section what these lines look like.

First, let's look at an octahedron. In figure II.3, I draw just the upper half of a regular octahedron, which is a pyramid. In **(a)** I label the four faces. In **(b)** I set up a coordinate system.

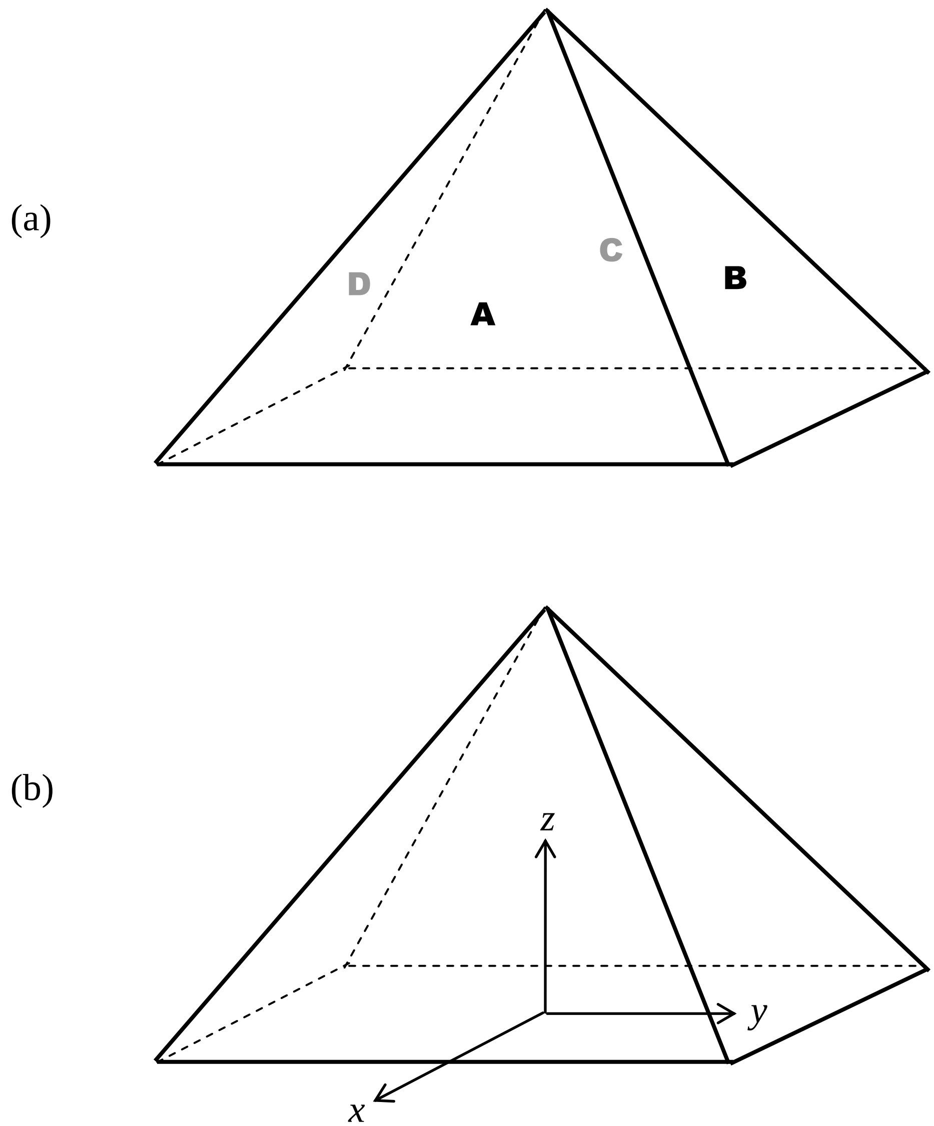


FIGURE II.3

I take the length of each edge to be 2. In that case, face **A** intersects the three coordinate axes at $x_0 = 1, \quad y_0 = \infty, \quad z_0 = \sqrt{2}$, and so, using equation 2.2.4, we find, for the equation to face **A**:

$$x + \frac{z}{\sqrt{2}} = 1. \tag{2.10.1}$$

Likewise, we find that the equation to face **B** is:

$$y + \frac{z}{\sqrt{2}} = 1. \tag{2.10.2}$$

Now let us make a plane slice through the meteorite, as indicated by the shaded ellipse in figure II.4. We'll suppose that the spherical angles of the normal to the slice are (θ, ϕ), and hence its direction cosines are $(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$.

(a)

(b)

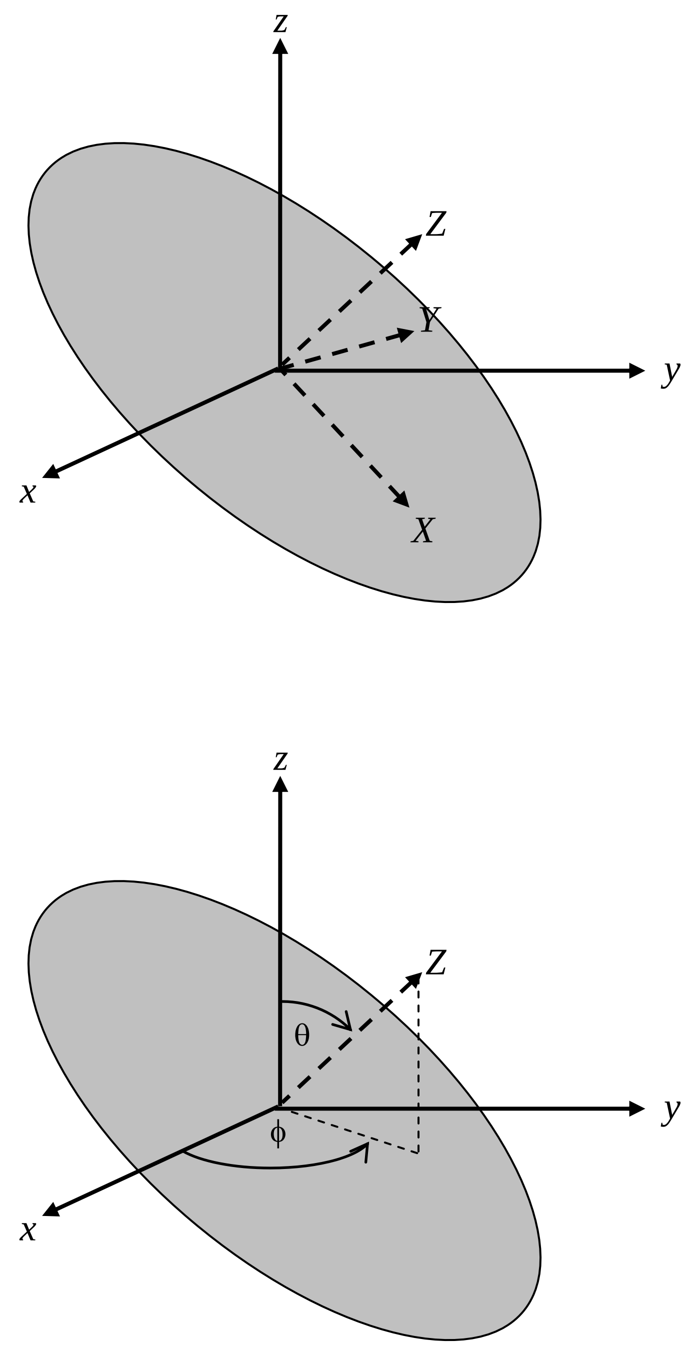


FIGURE II.4

The relations between the *xyz* and *XYZ* coordinate systems are:

$$x = X\cos\theta\cos\phi - Y\sin\phi + Z\sin\theta\cos\phi \qquad 2.10.3$$

$$y = X\cos\theta\sin\phi - Y\cos\phi + Z\sin\theta\sin\phi \qquad 2.10.4$$

$$z = -X\sin\theta\cos\phi + Z\cos\theta. \qquad 2.10.5$$

Face **A** of the octahedron intersects the *XY*-plane in a line whose equation is found by substitution of expressions 2.10.3, 2.10.4, 2.10.5 into equation 2.10.1 and then putting $Z = 0$, resulting in

$$X\left(\cos\theta\cos\phi - \tfrac{1}{\sqrt{2}}\sin\theta\right) - Y\sin\phi = 1. \qquad 2.10.6$$

This is one of the lines in the Widmanstätten pattern.

Likewise face **B** of the octahedron intersects the *XY*-plane in the line

$$X\left(\cos\theta\sin\phi - \tfrac{1}{\sqrt{2}}\sin\theta\right) - Y\cos\phi = 1. \qquad 2.10.7$$

The slopes of these two lines are, respectively,

$$m_{\mathbf{A}} = \frac{\cos\theta}{\tan\phi} - \frac{\sin\theta}{\sqrt{2}\sin\phi} \qquad 2.10.8$$

and $$m_{\mathbf{B}} = \cos\theta\tan\phi - \frac{\sin\theta}{\sqrt{2}\cos\phi}. \qquad 2.10.9$$

For example, suppose that the orientation of the slice is given by $\theta = 25^\circ$, $\phi = 15^\circ$, then the slopes of these two lines are:

$$m_{\mathbf{A}} = 2.2277, \qquad m_{\mathbf{B}} = -0.0665 \qquad 2.10.10$$

and so the angle between these two lines in the Widmanstätten pattern is <u>69º.6</u> .

Here is a photograph of the Widmanstätten lines in a slice of the Gibeon octahedrite from the collection of David Balam.

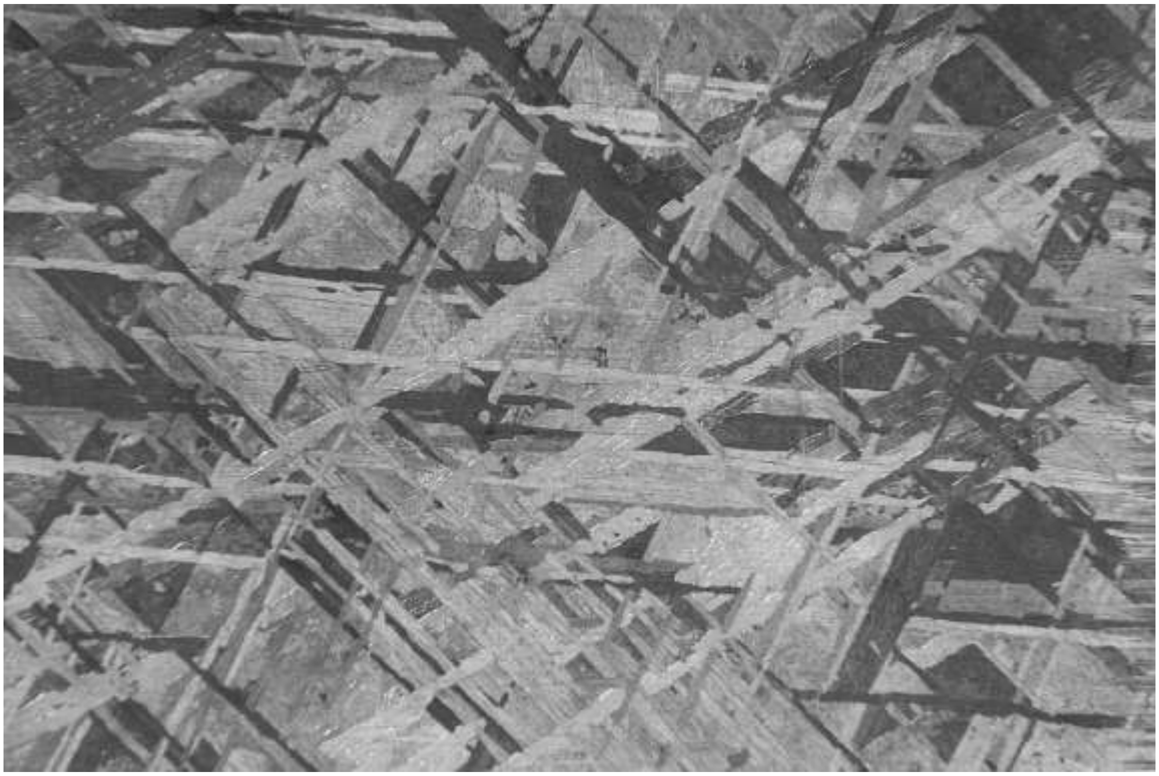


FIGURE II.4

For further details on this problem - including the inverse problem of determining θ and ϕ from measurements of the angles in the Widmanstätten pattern - see *Meteoritics and Planetary Science* **54**, 2977 (2019).

2.11. *An Exploding Meteoroid*

For our third and last example, we have a meteoroid hurtling through Earth's atmosphere. The surface becomes exceedingly hot, while the interior remains at the low temperature of outer space. Consequently there are enormous thermomechanical stresses within the meteoroid, which suddenly explodes in a terminal burst, which we shall treat as a point source of sound. The sound from this terminal burst is heard by three observers (or by three seismographs, which can detect sonic waves) on the ground. The sound arrives at the three observers at different times, because of the different distances of the observers from the terminal burst. We suppose that the three observers record the times when they hear the sound as well as the time when they saw the visual flash. The object is to determine the position of the meteoroid at the instant of the terminal burst. For the purposes of this example, we shall suppose that the atmosphere is isothermal, and hence that the speed of sound is the same everywhere. We also suppose that the Earth is flat except from the little hills on which the observers might be situated.

Let us set up a cartesian coordinate system O*xyz*. O is some point on the ground, and the *x*-, *y*- and *z*-axes point respectively to east, north and up. The coordinates of the three observers are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and the coordinates of the terminal burst are (x, y, z). The distances of the observers from the terminal burst are respectively d_1, d_2, d_3. If we suppose that each observer recorded the time of the visual flash and the time when he/she/it heard the bang, and that the speed of sound is known, then d_1, d_2, d_3 are all known. The terminal burst is at the centre of each of the three spheres of radii d_1, d_2, d_3. The equations of these spheres are

$$(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2 = d_1^2 \tag{2.11.1}$$

$$(x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2 = d_2^2 \tag{2.11.2}$$

$$(x_3 - x)^2 + (y_3 - y)^2 + (z_3 - z)^2 = d_3^2 \tag{2.11.3}$$

All that has to be done then is to solve these three equations for the three unknowns *x*, *y*, *z*. For example, if

$$\begin{aligned} x_1 &= 17, & y_1 &= 2, & z_1 &= 0.2, & d_1 &= 33.99 \\ x_2 &= 4, & y_2 &= 42, & z_2 &= 0.3, & d_2 &= 23.10 \\ x_3 &= 31, & y_3 &= 16, & z_3 &= 0.4, & d_3 &= 24.09 \end{aligned}$$

(all distances in km), then the solution to the three equations is

$$x = 20.39, \quad y = 32.95, \quad z = 13.83 \text{ km.}$$

The three equations are, of course, quadratic equations, and the practical solution of three quadratic equations in three unknowns requires a little computational experience

Further details of this problem are to be found in *Meteoritics and Planetary Science* **34**, 571 (1999).