CHAPTER 1
INTRODUCTION

1.1 *Quadric Surfaces*

These notes will be discussing the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0. \qquad 1.1.1$$

That is to say, the general second degree equation in three variables. It is assumed that the reader is already familiar with the geometry of the conic sections and the corresponding equation in two variables, as discussed for example, in Chapter 2 of my notes on Celestial Mechanics, astrowww.phys.uvic.ca/~tatum/celmechs.html If you are not familiar with the properties of the conic sections, and the corresponding equation in two variables, this set of notes may not be for you just yet.

Equation 1.1.1 describes a surface in three-dimensional space. Depending on the coefficients, the surface will usually be an *ellipsoid* (of which *spheroids* and *spheres* are special cases) or a *paraboloid* or a *hyperboloid*, though there are a few other possibilities, such as a *cone* or a *cylinder*, or one or two *planes*. I shall refer to the several possible surfaces represented by equation 1.1 as *quadric surfaces*, or occasionally for brevity if somewhat ungrammatically as just a *quadric.* If the coefficients of all the quadratic terms are zero. we are left with just $2ux + 2vy + 2wz + d = 0$, which is a single plane. We'll discuss planes in Chapter 2.

If you rotate an ellipse (Figure I.1 below) about its *major axis*, you generate a *prolate spheroid*. A rugger ball is an example. If you rotate an ellipse about its *minor axis*, you generate an *oblate spheroid*. The figure of the Earth (a sphere slightly flattened at the poles) is an example. If you have a figure in which three orthogonal sections are all ellipses (such a figure cannot be generated by rotating an ellipse), you have a *triaxial ellipsoid.*

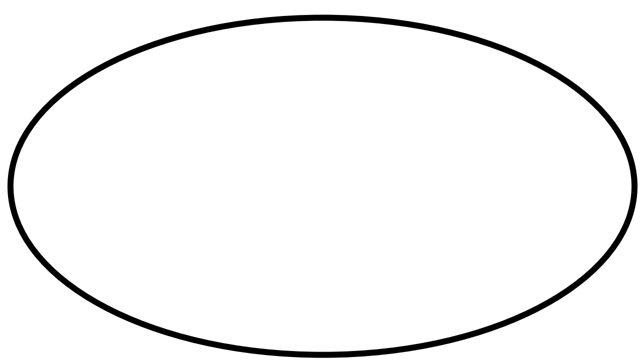

FIGURE I.1

If you rotate a parabola (Figure I.2 below) about its symmetry axis, you get what is commonly called a *paraboloid*, but which is in fact a special paraboloid which we should call a *circular paraboloid.* An example would be a reflecting telescope mirror. Another example would be the surface of a cup of tea when you have just stirred it. If you have a vat of molten glass or fused quartz which you rotate, its surface will take up the shape of a circular paraboloid, and if you let the glass cool and solidify while still rotating the vat, you have the makings of a fine paraboloidal telescope mirror.

It is quite possible to have a paraboloid whose cross-sections perpendicular to its symmetry axis are ellipses rather than circles, although these cannot be generated by rotating a parabola. This would be an *elliptical paraboloid*.

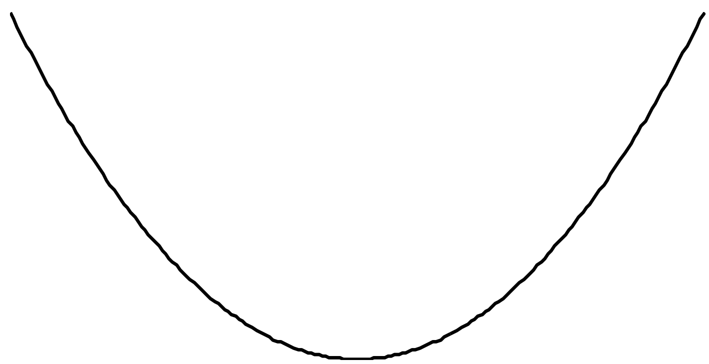

FIGURE I.2

If you rotate a hyperbola about its transverse axis (i.e. the horizontal axis in the Figure I.3 below), you generate a circular *hyperboloid of two sheets*. If you rotate it about its other axis (the vertical axis), you generate a circular *hyperboloid of one sheet.* The cross-sections of a hyperboloid need not be circular - they might be ellipses, although of course you cannot generate elliptical hyperboloids merely by rotating a hyperbola.

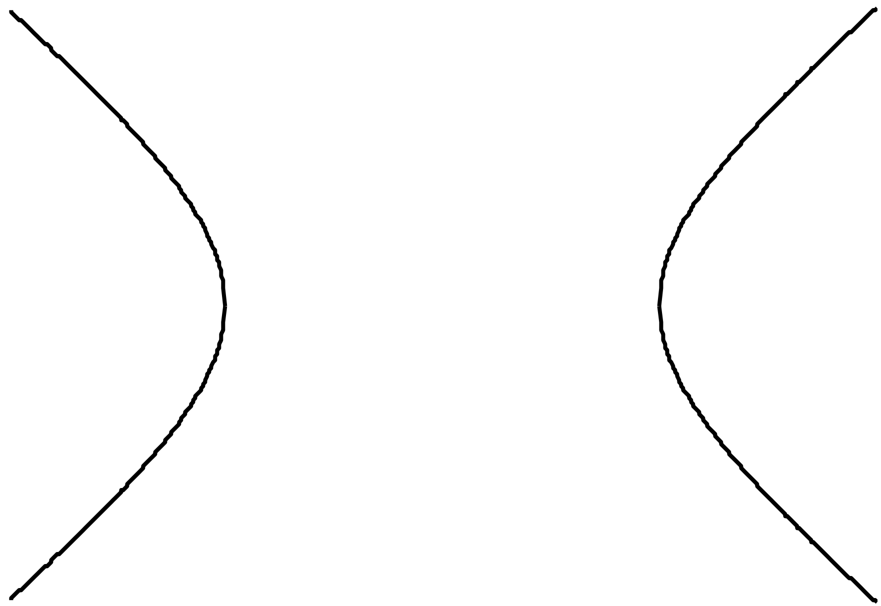

FIGURE I.3

Let us get back to equation 1.1.1, repeated here for convenience:

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0. \qquad 1.1.1$$

Under some circumstances (to be discussed in Section 5.2 of Chapter 5) it might be possible to translate (without rotation) the coordinate axes to a position such that equation 1.1.1, when referred to the new axes, takes the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d = 0 \qquad 1.1.2$$

Notice now that if you reverse the signs of *x* and *y* and *z* you change nothing, and the equation represents a figure that has a *centre of symmetry*, which coincides with the origin of the new coordinate axes. The figure is symmetric with reflection through its centre. Such a figure is a *central quadric*. That is, it is an ellipsoid or a hyperboloid or a cone - but not a paraboloid or a pair of planes.

It might now even be possible to *rotate* the axes of coordinates so that the equation takes an even simpler form:

$$ax^2 + by^2 + cz^2 = \text{constant}. \qquad 1.1.3$$

If we can succeed in doing this, we shall have found a coordinate system such that the axes of coordinates coincide with the symmetry axes of the quadric surface.

1.2 *Direction Cosines*

We shall have frequent need to describe the *orientation* of a line in three-dimensional space. This may easily be done by specifying the familiar meridional and azimuthal angles θ and ϕ of a spherical coordinate system (see Figure I.4a). θ is in the range 0º to 180º, and ϕ is in the range 0º to 360º. Equally frequently in these notes we shall use the three angles α, β, γ that the line makes with the *x*-, *y*-, *z*-axes (see Figure I.4b), or the *cosines* of these angles, $\cos\alpha$, $\cos\beta$, $\cos\gamma$. (The angle γ is identical with the angle θ.) These three cosines are known as the *direction cosines*. They are commonly denoted by l, m, n. It will not have escaped the reader that, from the theorem of Pythagoras, they are related by

$$l^2 + m^2 + n^2 = 1 \qquad 1.2.1$$

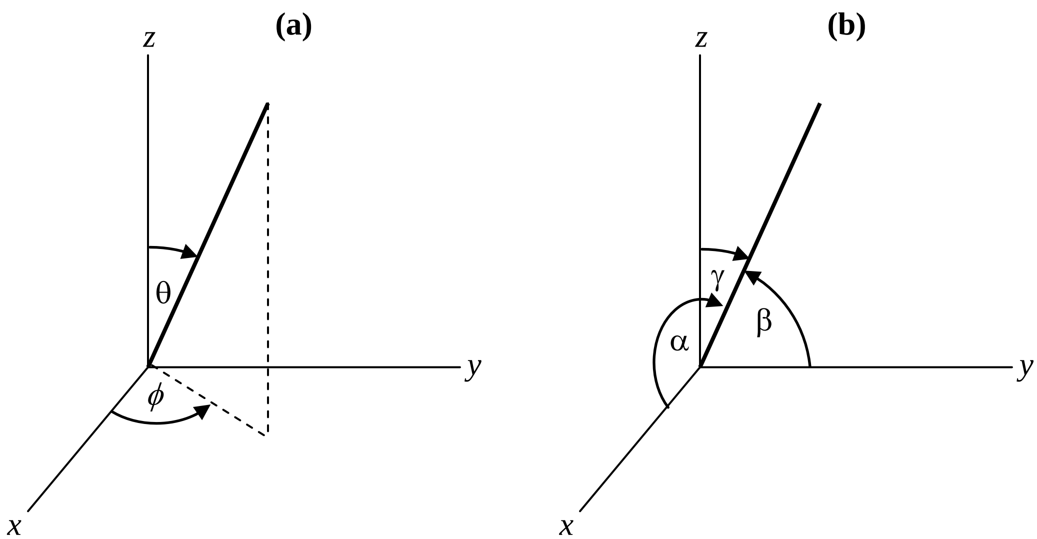

FIGURE I.4

The direction cosines are related to the spherical angles by the (we hope familiar) relations

$$l = \sin\theta \cos\phi \qquad 1.2.2$$
$$m = \sin\theta \sin\phi \qquad 1.2.3$$

$$n = \cos\theta \qquad 1.2.4$$

Example. $l = 0.2345$ and $m = 0.6789$. What are the angles α, β, γ?
Beware! This is not as straightforward as it sounds! First we'll use equation 1.2.1 to calculate n. Then we'll just calculate θ from equation 1.2.4. What could be easier? We obtain:

$l = 0.2345$	$m = 0.6789$	$n = \pm 0.6958$
α	β	γ
76º.44	47º.24	45º.91
283º.56	312º.76	314º.09
		134º.09
		225º.91

We find two possible answers for α, two for β, and four for γ – a total of 16 possible combinations! Which one do we want? We are reminded here that quadrant problems are among the most frustrating and frequent in trigonometry (and in celestial mechanics), and they cannot be ignored. We must never, ever forget that a square root has two solutions, and an inverse trigonometric function has two solutions between 0 and 360 degrees.

First: White it is true that *l, m, n* are related through equation 1.4, this does not excuse us from specifying all three. In setting the question, I could have specified that $n = +0.6958$. Second, the range of each of α, β, γ is from 0º to 180º. Assuming that I had specified that $n = +0.6958$, the answer to the problem is

$$\alpha = 76^{\circ}.44 \qquad \beta = 47^{\circ}.24 \qquad \gamma = 45^{\circ}.91$$

It is, however, permissible to change the signs of all the direction cosines. For example,
$l = 0.4623$, $m = -0.2948$ $n = 0.8329$ yields
$\alpha = 62^{\circ}$ $\beta = 107^{\circ}$ $\gamma = 34^{\circ}$
while
$l = -0.4623$, $m = 0.2948$ $n = -0.8329$ yields
$\alpha = 118^{\circ}$ $\beta = 73^{\circ}$ $\gamma = 146^{\circ}$

These two orientations are identical, except that they are expressed in opposite octants.

The number triplet (l, m, n) tells the orientation of a line, and it can be regarded as a *vector*, with components *l, m, n*. And, since the components obey equation 1.2.1, the set of direction cosines is a *unit vector.* If two lines with direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2) are at right angles to each other, their dot product $l_1 l_2 + m_1 m_2 + n_1 n_2$ is zero.

A set of three numbers that are *proportional* to the direction cosines - that is to say cl, cm, cn - are *direction ratios.* A set of direction ratios tells us the orientation of a line just as well as does a set of direction cosines, and the set constitutes a vector, but not a unit vector.

The distance r from the origin to the point (a,b,c) is $\sqrt{a^2 + b^2 + c^2}$, and the direction cosines of the line joining the origin to the point (a,b,c) are $(a/r,\ b/r,\ c/r)$. The triplet (a,b,c) is, then, a set of direction ratios of the line joining the origin to the point (a,b,c). If a line has direction ratios (a,b,c), its direction cosines are

$$\left(\frac{a}{\sqrt{a^2+b^2+c^2}},\ \frac{b}{\sqrt{a^2+b^2+c^2}},\ \frac{c}{\sqrt{a^2+b^2+c^2}}\right). \qquad 1.2.5$$

If we have two points (a_1,b_1,c_1), (a_2,b_2,c_2), then the line joining them has direction ratios $(a_1-a_2,\ b_1-b_2,\ c_1-c_2)$ and hence direction cosines

$$(l,m,n) = \left(\frac{a_1-a_2}{\sqrt{a^2+b^2+c^2}},\ \frac{b_1-b_2}{\sqrt{a^2+b^2+c^2}},\ \frac{c_1-c_2}{\sqrt{a^2+b^2+c^2}}\right). \qquad 1.2.6$$

The spherical angles are found from

$$\theta = \cos^{-1} n, \quad \phi = \tan^{-1}(m/l) \qquad 1.2.7$$

1.3. *Rotation of Coordinate Axes*

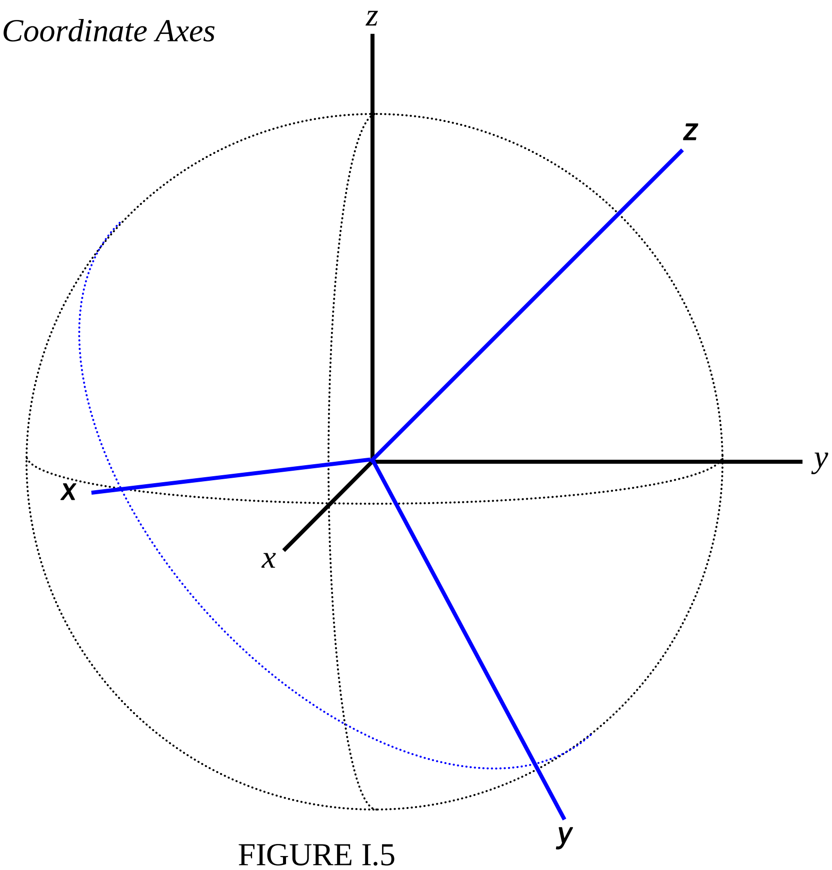

FIGURE I.5

Refer to figure I.5. We draw, in black, a set of coordinate axes, *xyz*. I have written *xyz* in *Times New Roman italic* font, and I shall refer to these axes as the "Roman" axes.

We shall often have to refer to another set of axes ***xyz***, which I have drawn in blue. I have written them in ***Franklin Gothic Book boldface italic*** font, and I shall refer to them as the "Franklin axes".

The coordinates in the two systems are related by

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} \qquad 1.2.8a,b$$

in which m_1 is the cosine of the angle between the ***x*** and *y* axes, etc., and (l_1, m_1, n_1) are the direction cosines of the ***x*** axis rerferred to the Roman system, etc. This is a *unit orthogonal* transformation.

We'll suppose that the spherical angles of the ***z*** axis, referred to the Roman system, are (θ, ϕ). From the usual relations between three-dimensional cartesian and spherical coordinates, these are related by:

$$(l_3, m_3, n_3) = (\sin\theta\cos\phi, \quad \sin\theta\sin\phi, \ \cos\theta) \qquad 1.2.9$$

The ***x*** axis is, of course, at right angles to the ***z*** axis (it is in the ***xy*** plane), but, further, I shall choose it, as in figure I.5, to be at right angles to the *z* axis (i.e. in the *xy* plane). Its direction cosines are (l_1, m_1, n_1). Referred to the Roman system, its meridional angle is 90º, and its azimuthal angle is 270º + ϕ. Hence:

$$(l_1, m_1, n_1) = (\sin\phi, \ -\cos\phi, \ 0) \qquad 1.2.10$$

An easy way of determining the direction cosines and spherical angles of the ***y*** is to recall that, in a unit orthogonal transform, every element is equal to its own cofactor. That is

$$l_2 = \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix}, \qquad m_2 = \begin{vmatrix} n_3 & n_1 \\ l_3 & l_1 \end{vmatrix}, \qquad n_2 = \begin{vmatrix} l_3 & l_1 \\ m_3 & m_1 \end{vmatrix}. \qquad 1.2.11$$

That is to say

$$(l_2, m_2, n_2) = (\cos\theta\cos\phi, \quad \cos\theta\sin\phi, \ -\sin\theta) \qquad 1.2.12$$

The meridional and azimuthal angles of the $\boldsymbol{y}$-axes are, respectively, 90º + θ and ϕ. The meridional angle is the angle between the z- and $\boldsymbol{y}$-axes, and the cosine of this angle is n_2. Thus we see again that n_2 is equal to $-\sin\theta$.

Thus the relations between the Roman and Franklin coordinates are

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sin\phi & -\cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad 1.2.13$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = \begin{pmatrix} \sin\phi & \cos\theta\cos\phi & \sin\theta\cos\phi \\ -\cos\phi & \cos\theta\sin\phi & \sin\theta\sin\phi \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} \qquad 1.2.14$$

We shall be making use of these later.