PREAMBLE

In my online notes on Celestial Mechanics astrowww.phys.uvic.ca/~tatum/celmechs.html I included a chapter on the conic sections. The three-dimensional equivalents of the conic sections - known as the quadric surfaces - didn't seem to have as much obvious connection to celestial mechanics, so they did not get a mention there. However, I am a lover of both conic sections and quadric surfaces, and the latter have been nagging at me - so here, for no particularly good reason, are a few notes on them.

These notes will include many numerical examples, some of which will involve extensive computation. It is *very strongly* recommended that, before you start reading any of these notes, you provide yourself with computer programs that will instantly carry out the following tasks for you:

Solve a quadratic equation.
Solve two simultaneous linear equations.
Solve three simultaneous linear equations.
Evaluate a 3×3 determinant.
Evaluate a 4×4 determinant.
Test a 3×3 matrix for unit orthogonality.

You will not enjoy these notes if you constantly have to stop to do tedious calculations (in which you are likely to make mistakes and have to start all over again). They will be far more enjoyable if, whenever you encounter any equation, you can instantly obtain the answer without interrupting your train of thought. It is expected that you will be reading these notes while sitting in front of your computer, instantly ready to do any of the above tasks.

I imagine that some or perhaps all of these processes might be found in various mathematical websites such as Wolfram Alpha. If you are experienced in programming you will probably find it at least as convenient to write your own programs. In this Preamble, I provide some guidance in programming these tasks.

In case you are thinking that this is a lot of work - the first task (solving a quadratic equation) is the most difficult to program. All other tasks listed above are easier to program. They may be longer, but are more straightforward.

There are lots and lots of equations, and there are sure to be a few typos. If you find any, please do let me know at jtatum at uvic dot ca For from being annoyed, I shall be very happy to have mistakes pointed out to me. Not only does it enable me to correct them, but it shows me that people are actually reading the stuff!

Solve a quadratic equation.

$$ax^2 + bx + c = 0$$

Calculate the discriminant: $d = b^2 - 4ac$

If $d > 0$ then: $s = \sqrt{d}$

$$x_1 = \frac{-b+s}{2a} \qquad x_2 = \frac{-b-s}{2a}$$

If $d = 0$ then $x = -\frac{b}{2a}$

If $d < 0$ then $s = \sqrt{-d}$

$$x_{\text{real}} = -\frac{b}{2a} \qquad x_{\text{im}} = \frac{s}{2a} \qquad x = x_{\text{real}} \pm i x_{\text{im}}$$

Solve two simultaneous linear equations.

The solutions to

$$ax + by + c = 0$$
$$dx + ey + f = 0$$

Calculate $g = bd - ae.$

Then $x = \frac{ce - bf}{g} \qquad y = \frac{af - cd}{g}$

Solve three simultaneous linear equations.

$$P_1 x + Q_1 y + R_1 z + S_1 = 0$$
$$P_2 x + Q_2 y + R_2 z + S_2 = 0$$
$$P_3 x + Q_3 y + R_3 z + S_3 = 0$$

Calculate:

$$a = P_1R_2 - R_1P_2$$
$$b = Q_1R_2 - R_1Q_2$$
$$c = S_1R_2 - R_1S_2$$
$$d = P_1R_3 - R_1P_3$$
$$e = Q_1R_3 - R_1Q_3$$
$$f = S_1R_3 - R_1S_3$$
$$g = bd - ae$$

Then $$x = \frac{ce - bf}{g} \qquad y = \frac{af - cd}{g} \qquad z = -\frac{P_1x + Q_1y + S_1}{R_1}$$

There are lots of other ways of doing it, but this is straightforward to program and works well as long as neither g nor R_1 are zero.

Evaluate a 3×3 determinant.

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = A_1(B_2C_3 - C_2B_3) + B_1(C_2A_3 - A_2C_3) + C_1(A_2B_3 - B_2A_3)$$

Evaluate a 4×4 determinant.

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}$$

Calculate the following quantities:

$$P_1 = B_2C_3D_4 + C_2D_3B_4 + D_2B_3C_4$$
$$P_2 = B_4C_3D_2 + C_4D_3B_2 + D_4B_3C_2$$
$$P = P_1 - P_2$$

$$Q_1 = A_2C_3D_4 + C_2D_3A_4 + D_2A_3C_4$$
$$Q_2 = A_4C_3D_2 + C_4D_3A_2 + D_4A_3C_2$$
$$Q = Q_1 - Q_2$$

$$R_1 = A_2B_3D_4 + B_2D_3A_4 + D_2A_3B_4$$
$$R_2 = A_4B_3D_2 + B_4D_3A_2 + D_4A_3B_2$$
$$R = R_1 - R_2$$

$$S_1 = A_2B_3C_4 + B_2C_3A_4 + C_2A_3B_4$$
$$S_2 = A_4B_3C_2 + B_4C_3A_2 + C_4A_3B_2$$
$$S = S_1 - S_2$$

Then: Determinant = $A_1P - B_1Q + C_1R - D_1S$

Test a 3×3 matrix for unit orthogonality

This may take slightly longer than the other tasks to program, and there may be a temptation not to program it. Do not skip it! This is a most important task to perform, and it is straightforward to program, and only slightly longer than the other tasks. While longer, it is not as difficult to program as solving a quadratic equation. Once programmed, the computer carries out the program instantly without any trouble. Also, not only does it determine whether the matrix is orthogonal, but if there are any mistakes in any of the elements, it shows where the mistake is.

The three columns or three rows of a unit orthogonal matrix represent the components of three unit orthogonal vectors. The length of each such vector must be unity, the scalar product of any two of them must be zero, and the vector product of any two of them must equal the third. This means that the matrix must satisfy the following conditions:

1. For any row or column, the sum of the squares of the elements must be unity.
2. The sum of the products of corresponding elements in any two rows or any two columns must be zero.
3. Every element must equal its cofactor.

A further test to be satisfied is that the determinant of the matrix must be ± 1. If it is +1, the unit orthogonal vectors form a right-handed system. If it is -1, they form a left-handed system.

Given the matrix $\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$, is it a unit orthogonal matrix?

Condition **1**. Calculate and display the following quantities. All should be 1.

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COL1=A11*A11+A21*A21+A31*A31
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COL2=A12*A12+A22*A22+A32*A32
COL3=A13*A13+A23*A23+A33*A33
ROW1=A11*A11+A12*A12+A13*A13
ROW2=A21*A21+A22*A22+A23*A23
ROW3=A31*A31+A32*A32+A33*A33

Condition **2**. Calculate and display the following quantities. All should be 0.

COLS12=A11*A12+A21*A22+A31*A32
COLS13=A11*A13+A21*A23+A31*A33
COLS23=A12*A13+A22*A23+A32*A33
ROWS12=A11*A21+A12*A22+A13*A23
ROWS13=A11*A31+A12*A32+A13*A33
ROWS23=A21*A31+A22*A32+A23*A33

Condition **3**. Calculate and display the following quantities. (Each such quantity is one of the elements of the matrix minus it cofactor). All should be 0.

C11=A11-(A22*A33-A32*A23)
C12=A12-(A23*A31-A33*A21)
C13=A13-(A21*A32-A31*A22)
C21=A21-(A32*A13-A12*A33)
C22=A22-(A11*A33-A31*A13)
C23=A23-(A31*A12-A11*A32)
C31=A31-(A12*A23-A22*A13)
C32=A32-(A21*A13-A11*A23)
C33=A33-(A11*A22-A21*A12)

Finally calculate and display the value of the determinant. It should be ± 1. If it is -1, the three axes form a left-handed system.

DET=A11*(A22*A33-A32*A23)+A12*(A23*A31-A33*A21)+A13*(A21*A32-A31*A22)