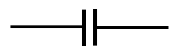
CHAPTER 5
CAPACITORS

5.1 *Introduction*

A capacitor consists of two metal plates separated by a nonconducting medium (known as the *dielectric medium* or simply the *dielectric*, or by a vacuum. It is represented by the electrical symbol

 .

Capacitors of one sort or another are included in almost any electronic device. Physically, there is a vast variety of shapes, sizes and construction, depending upon their particular application. This chapter, however, is not primarily concerned with real, practical capacitors and how they are made and what they are used for, although a brief section at the end of the chapter will discuss this. In addition to their practical uses in electronic circuits, capacitors are very useful to professors for torturing students during exams, and, more importantly, for helping students to understand the concepts of and the relationships between electric fields **E** and **D**, potential difference, permittivity, energy, and so on. The capacitors in this chapter are, for the most part, imaginary academic concepts useful largely for pedagogical purposes. Need the electronics technician or electronics engineer spend time on these academic capacitors, apparently so far removed from the real devices to be found in electronic equipment? The answer is surely and decidedly <u>*yes*</u> – more than anyone else, the practical technician or engineer must thoroughly understand the basic concepts of electricity before even starting with real electronic devices.

If a potential difference is maintained across the two plates of a capacitor (for example, by connecting the plates across the poles of a battery) a charge $+Q$ will be stored on one plate and $-Q$ on the other. The ratio of the charge stored on the plates to the potential difference V across them is called the *capacitance* C of the capacitor. Thus:

$$Q = CV. \qquad 5.1.1$$

If, when the potential difference is one volt, the charge stored is one coulomb, the capacitance is one *farad*, F. Thus, a farad is a coulomb per volt. It should be mentioned here that, in practical terms, a farad is a very large unit of capacitance, and most capacitors have capacitances of the order of microfarads, μF.

The dimensions of capacitance are $\dfrac{\mathrm{Q}}{\mathrm{ML^2T^{-2}Q^{-1}}} = \mathrm{M^{-1}L^{-2}T^2Q^2}$.

It might be remarked that, in older books, a capacitor was called a "condenser", and its capacitance was called its "capacity". Thus what we would now call the "capacitance of a capacitor" was formerly called the "capacity of a condenser".

In the highly idealized capacitors of this chapter, the linear dimensions of the plates (length and breadth, or diameter) are supposed to be very much larger than the separation between them. This in fact is nearly always the case in real capacitors, too, though perhaps not necessarily for the same reason. In real capacitors, the distance between the plates is small so that the capacitance is as large as possible. In the imaginary capacitors of this chapter, I want the separation to be small so that the electric field between the plates is uniform. Thus the capacitors I shall be discussing are mostly like figure V.1, where I have indicated, in blue, the electric field between the plates:

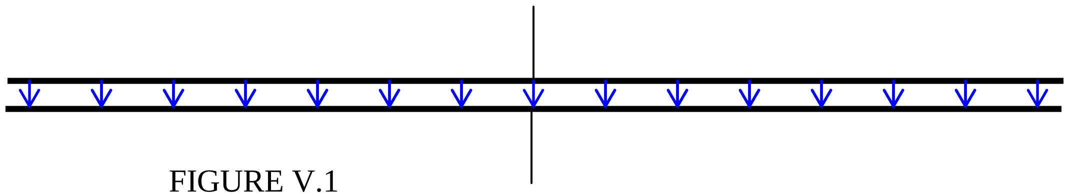
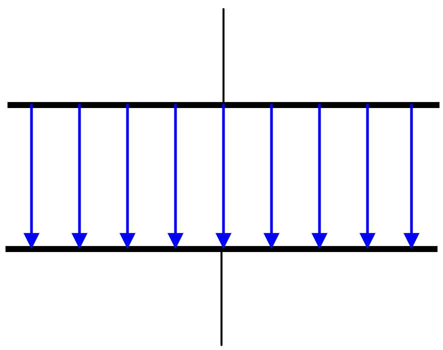


FIGURE V.1

However, I shall not always draw them like this, because it is rather difficult to see what is going on inside the capacitor. I shall usually much exaggerate the scale in one direction, so that my drawings will look more like this:

FIGURE V.2

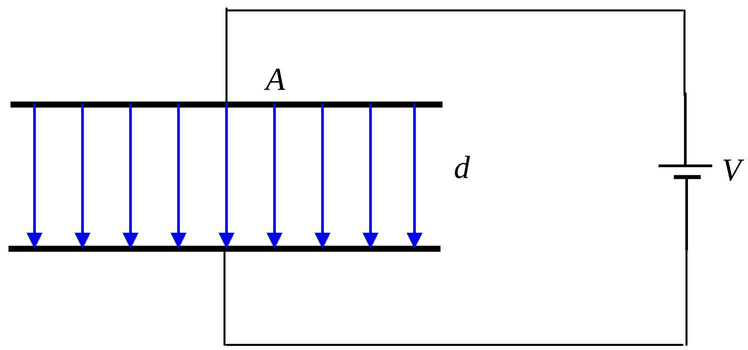


If the separation were really as large as this, the field would not be nearly as uniform as indicated; the electric field lines would greatly bulge outwards near the edges of the plates.

In the next few sections we are going to derive formulas for the capacitances of various capacitors of simple geometric shapes.

5.2 *Plane Parallel Capacitor*

FIGURE V.3



We have a capacitor whose plates are each of area A, separation d, and the medium between the plates has permittivity ε. It is connected to a battery of EMF V, so the potential difference across the plates is V. The electric field between the plates is $E = V/d$, and therefore $D = \varepsilon V/d$. The total D-flux arising from the positive plate is DA, and, by Gauss's law, this must equal Q, the charge on the plate.

Thus $Q = \varepsilon AV/d$, and therefore the capacitance is

$$C = \frac{\varepsilon A}{d}. \qquad 5.2.1$$

Verify that this is dimensionally correct, and note how the capacitance depends upon ε, A and d.

In Section 1.5 we gave the SI units of permittivity as $\mathrm{C^2\,N^{-1}\,m^{-2}}$. Equation 5.2.1 shows that a more convenient SI unit for permittivity is $\mathrm{F\,m^{-1}}$, or farads per metre.

Question: If the separation of the plates is not small, so that the electric field is not uniform, and the field lines bulge outwards at the edge, will the capacitance be less than or greater than $\varepsilon A/d$?

5.3 *Coaxial Cylindrical Capacitor*

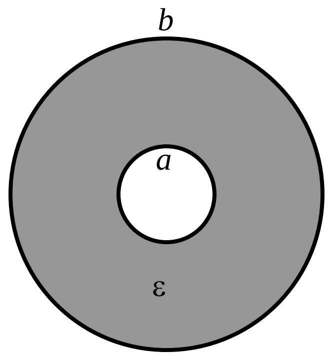


FIGURE V.4

The radii of the inner and outer cylinders are *a* and *b*, and the permittivity between them is ε. Suppose that the two cylinders are connected to a battery so that the potential difference between them is *V*, and the charge per unit length on the inner cylinder is +λ C m^{-1}, and on the outer cylinder is −λ C m^{-1}. We have seen (Subsection 2.2.3) that the potential difference between the cylinders under such circumstances is $\frac{\lambda}{2\pi\varepsilon}\ln(b/a)$. Therefore the capacitance per unit length, C', is

$$C' = \frac{2\pi\varepsilon}{\ln(b/a)}. \qquad 5.3.1$$

This is by no means solely of academic interest. The capacitance per unit length of coaxial cable ("coax") is an important property of the cable, and this is the formula used to calculate it.

5.4 *Concentric Spherical Capacitor*

Unlike the coaxial cylindrical capacitor, I don't know of any very obvious practical application, nor quite how you would construct one and connect the two spheres to a battery, but let's go ahead all the same. Figure V.4 will do just as well for this one.

The two spheres are of inner and outer radii *a* and *b*, with a potential difference *V* between them, with charges +*Q* and −*Q* on the inner and outer spheres respectively. The potential difference between the two spheres is then $\frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$, and so the capacitance is

$$C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}. \qquad 5.4.1$$

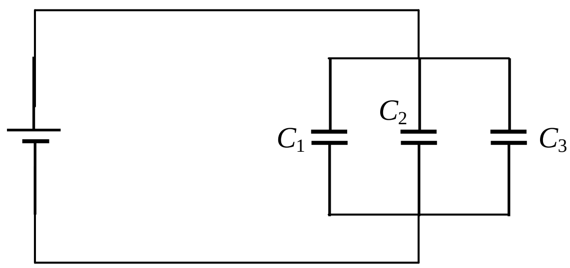
If $b \to \infty$, we obtain for the capacitance of an isolated sphere of radius *a*:

$$C = 4\pi\epsilon a. \qquad 5.4.2$$

Exercise: Calculate the capacitance of planet Earth, of radius 6.371×10^3 km, suspended in free space. I make it 709 μF - which may be a bit smaller than you were expecting.

5.5 *Capacitors in Parallel*

FIGURE V.5



The potential difference is the same across each, and the total charge is the sum of the charges on the individual capacitor. Therefore:

$$C = C_1 + C_2 + C_3. \qquad 5.5.1$$

5.6 *Capacitors in Series*

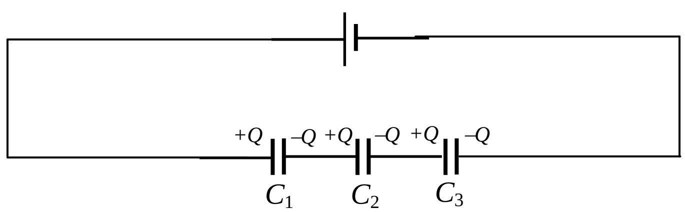


FIGURE V.6

The charge is the same on each, and the potential difference across the system is the sum of the potential differences across the individual capacitances. Hence

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}. \qquad 5.6.1$$

5.7 *Delta-Star Transform*

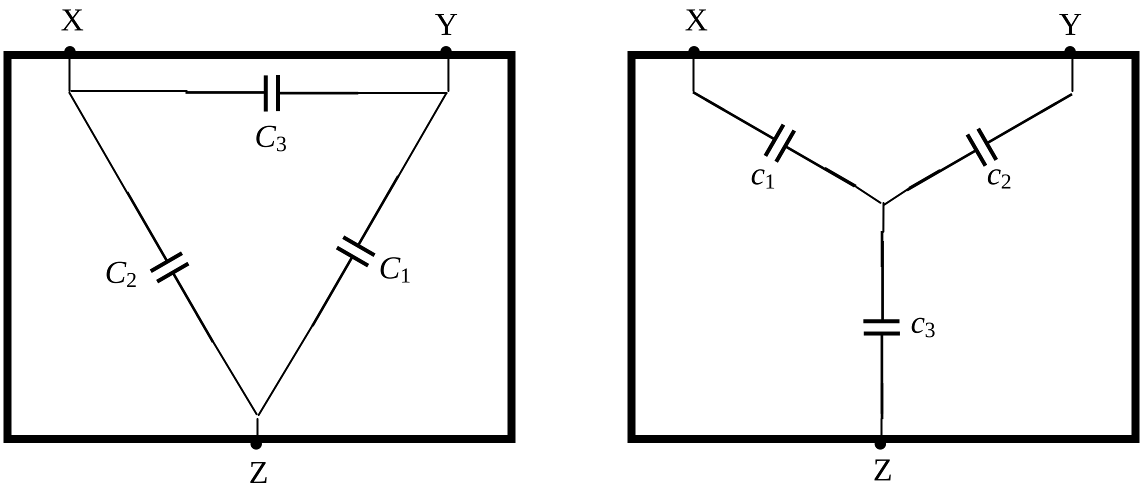


FIGURE V.7

As we did with resistors in Section 4.12, we can make a delta-star transform with capacitors. I leave it to the reader to show that the capacitance between any two terminals in the left hand box is the same as the capacitance between the corresponding two terminals in the right hand box provided that

$$c_1 = \frac{C_2C_3 + C_3C_1 + C_1C_2}{C_1}, \qquad 5.7.1$$

$$c_2 = \frac{C_2C_3 + C_3C_1 + C_1C_2}{C_2} \qquad 5.7.2$$

and

$$c_3 = \frac{C_2C_3 + C_3C_1 + C_1C_2}{C_3}. \qquad 5.7.3$$

The converse relations are

$$C_1 = \frac{c_2c_3}{c_1 + c_2 + c_3}, \qquad 5.7.4$$

$$C_2 = \frac{c_3c_1}{c_1 + c_2 + c_3} \qquad 5.7.5$$

and

$$C_3 = \frac{c_1c_2}{c_1 + c_2 + c_3}. \qquad 5.7.6$$

For example, just for fun, what is the capacitance between points A and B in figure V.8, in which I have marked the individual capacitances in microfarads?

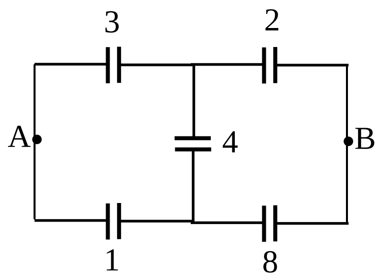


FIGURE V.8

The first three capacitors are connected in delta. Replace them by their equivalent star configuration. After that it should be straightforward. I make the answer 2.515 μF.

5.8 *Kirchhoff's Rules*

We can even adapt Kirchhoff's rules to deal with capacitors. Thus, connect a 24 V battery across the circuit of figure V.8 – see figure V.9

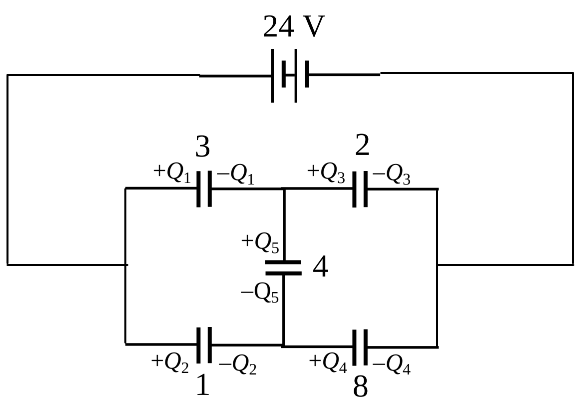


FIGURE V.9

Calculate the charge held in each capacitor. We can proceed in a manner very similar to how we did it in Chapter 4, applying the capacitance equivalent of Kirchhoff's second rule to three closed circuits, and then making up the five necessary equations by applying "Kirchhoff's first rule" to two points. Thus:

$$24 - \frac{Q_2}{3} - \frac{Q_3}{2} = 0\,, \qquad 5.8.1$$

$$24 - Q_2 - \frac{Q_4}{8} = 0\,, \qquad 5.8.2$$

$$\frac{Q_1}{3} - Q_2 + \frac{Q_5}{4} = 0, \qquad 5.8.3$$

$$Q_1 = Q_3 + Q_5\,, \qquad 5.8.4$$

and

$$Q_4 = Q_2 + Q_5\,. \qquad 5.8.5$$

I make the solutions

$$Q_1 = +41.35\,\mu\text{C},\;\; Q_2 = +19.01\,\mu\text{C},\;\; Q_3 = +20.44\,\mu\text{C},\;\; Q_4 = +39.92\,\mu\text{C},\;\; Q_5 = +20.91\,\mu\text{C}.$$

5.9 *Problem for a Rainy Day*

Another problem to while away a rainy Sunday afternoon would be to replace each of the resistors in the cube of subsection 4.14.1 with capacitors each of capacitance c. What is the total capacitance across opposite corners of the cube? I would start by supposing that the cube holds a net charge of $6Q$, and I would then ask myself what is the charge held in each of the individual capacitors. And I would then follow the potential drop from one corner of the cube to the opposite corner. I make the answer for the effective capacitance of the entire cube $1.2c$.

5.10 *Energy Stored in a Capacitor*

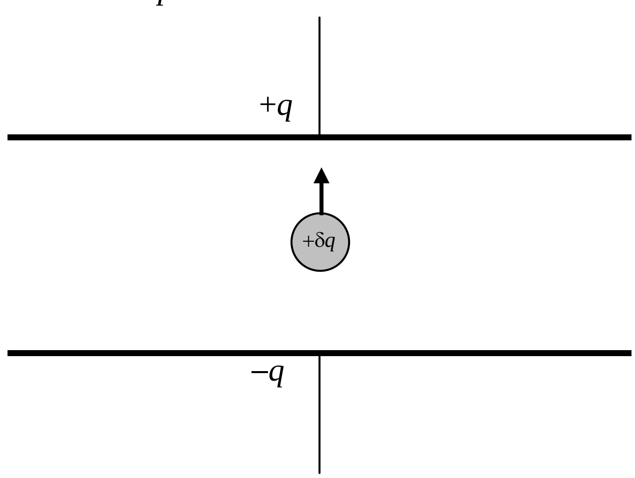


FIGURE V.10

Let us imagine (figure V.10) that we have a capacitor of capacitance C which, at some time, has a charge of $+q$ on one plate and a charge of $-q$ on the other plate. The potential difference across the plates is then q/C. Let us now take a charge of $+\delta q$ from the bottom plate (the negative one) and move it up to the top plate. We evidently have to do work to do this, in the amount of $\frac{q}{C}\delta q$. The total work required, then, starting with the plates completely uncharged until we have transferred a charge Q from one plate to the other is $\frac{1}{C}\int_0^Q q\,dq = Q^2/(2C)$. This is, then, the energy $\mathcal{E}$ stored in the capacitor, and, by application of $Q = CV$ it can also be written $\mathcal{E} = \frac{1}{2}QV$, or, more usually,

$$\mathcal{E} = \tfrac{1}{2}CV^2. \qquad 5.10.1$$

Verify that this has the correct dimensions for energy. Also, think about how many expressions for energy you know that are of the form $\frac{1}{2}ab^2$. There are more to come.

The symbol E is becoming rather over-worked. At present I am using the following:

E = magnitude of the electric field
$\mathbf{E}$ = electric field as a vector
$\mathcal{E}$ = electromotive force
$\mathcal{E}$ = energy

Sorry about that!

5.11 *Energy Stored in an Electric Field*

Recall that we are assuming that the separation between the plates is small compared with their linear dimensions and that therefore the electric field is uniform between the plates.

The capacitance is $C = \epsilon A/d$, and the potential difference between the plates is Ed, where E is the electric field and d is the distance between the plates. Thus the energy stored in the capacitor is $\frac{1}{2}\epsilon E^2 Ad$. The volume of the dielectric (insulating) material between the plates is Ad, and therefore we find the following expression for the *energy stored per unit volume in a dielectric material in which there is an electric field*:

$$\tfrac{1}{2}\epsilon E^2.$$

Verify that this has the correct dimensions for energy per unit volume.

If the space between the plates is a vacuum, we have the following expression for the energy stored ber unit volume in the electric field

$$\tfrac{1}{2}\varepsilon_0 E^2$$

- even though there is absolutely nothing other than energy in the space. Think about that!

I mentioned in Section 1.7 that in an *anisotropic medium* **D** and **E** are not parallel, the permittivity then being a tensor quantity. In that case the correct expression for the energy per unit volume in an electric field is $\tfrac{1}{2}\mathbf{D}\bullet\mathbf{E}$.

5.12 *Force Between the Plates of a Plane Parallel Plate Capacitor*

We imagine a capacitor with a charge $+Q$ on one plate and $-Q$ on the other, and initially the plates are almost, but not quite, touching. There is a force F between the plates. Now we gradually pull the plates apart (but the separation remains small enough that it is still small compared with the linear dimensions of the plates and we can maintain our approximation of a uniform field between the plates, and so the force remains F as we separate them). The work done in separating the plates from near zero to d is Fd, and this must then equal the energy stored in the capacitor, $\tfrac{1}{2}QV$. The electric field between the plates is $E = V/d$, so we find for the force between the plates

$$F = \tfrac{1}{2}QE. \qquad 5.12.1$$

We can now do an interesting imaginary experiment, just to see that we understand the various concepts. Let us imagine that we have a capacitor in which the plates are horizontal; the lower plate is fixed, while the upper plate is suspended above it from a spring of force constant k. We connect a battery across the plates, so the plates will attract each other. The upper plate will move down, but only so far, because the electrical attraction between the plates is countered by the tension in the spring. Calculate the equilibrium separation x between the plates as a function of the applied voltage V. (Horrid word! We don't say "metreage" for length, "kilogrammage" for mass or "secondage" for time – so why do we say "voltage" for potential difference and "acreage" for area? Ugh!) We should be able to use our invention as a voltmeter – it even has an infinite resistance!

Refer to figure V.11.

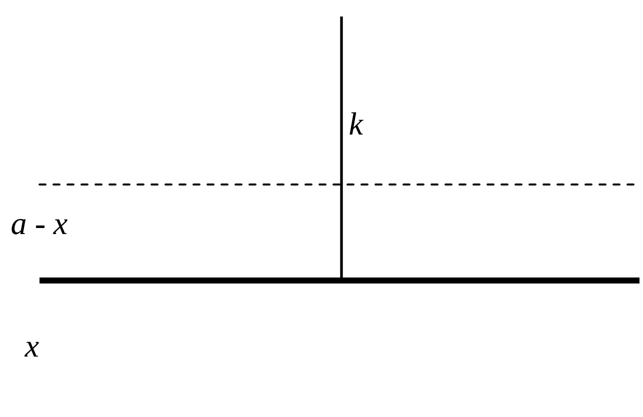


FIGURE V.11

We'll suppose that the separation when the potential difference is zero is a, and the separation when the potential difference is V is x, at which time the spring has been extended by a length $a - x$.

The electrical force between the plates is $\frac{1}{2}QE$. Now $Q = CV = \frac{\varepsilon_0 AV}{x}$ and $E = \frac{V}{x}$, so the force between the plates is $\frac{\varepsilon_0 AV^2}{2x^2}$. Here A is the area of each plate and it is assumed that the experiment is done in air, whose permittivity is very close to ε_0. The tension in the stretched spring is $k(a - x)$, so equating the two forces gives us

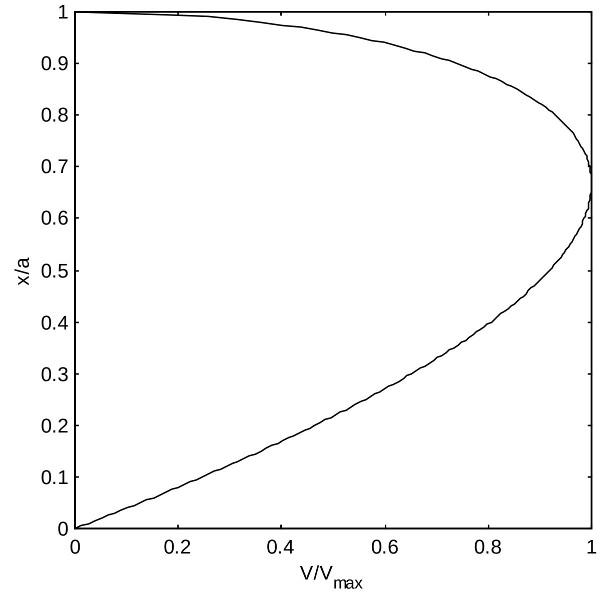
$$V^2 = \frac{2kx^2(a-x)}{\varepsilon_0 A}. \qquad 5.12.2$$

Calculus shows [do it! – just differentiate $x^2(1 - x)$] that V has a maximum value of $V_{\text{max}} = \sqrt{\frac{8ka^3}{27\varepsilon_0 A}}$ for a separation $x = \frac{2}{3}a$. If we express the potential difference in units of V_{max} and the separation in units of a, equation 5.12.2 becomes

$$V^2 = \frac{27x^2(1-x)}{4}. \qquad 5.12.3$$

In figure V.12 I have plotted the separation as a function of the potential difference.

FIGURE V.10



As expected, the potential difference is zero when the separation is 0 or 1 (and therefore you would expect it to go through a maximum for some intermediate separation).

We see that for $V < V_{\text{max}}$ there are *two* equilibrium positions. For example, if $V = 0.8$, show that $x = 0.396\ 305$ or $0.876\ 617$. The question also arises – what happens if you apply across the plates a potential difference that is *greater than* V_{max}?

Further insight can be obtained from energy considerations. The potential energy of the system is the work done in moving the upper plate from $x = a$ to $x = x$ while the potential difference is V:

$$E = \frac{\varepsilon_0 A V^2}{2a} - \frac{\varepsilon_0 A V^2}{2x} + \tfrac{1}{2}k(a-x)^2 . \qquad 5.12.4$$

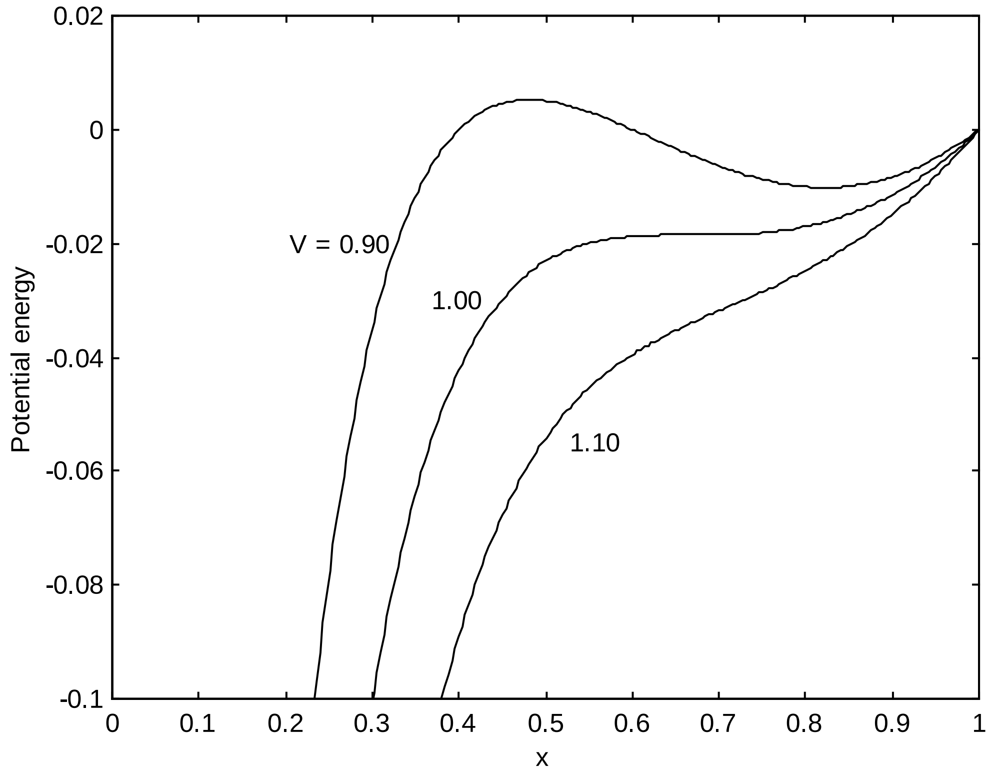
You may need to refer to Section 5.15 to be sure that we have got this right.

If we express V in units of V_{max} , x in units of a and E in units of ka^2, this becomes

$$E = \tfrac{4}{27}V^2(1 - 1/x) + \tfrac{1}{2}(1 - x)^2 . \qquad 5.12.5$$

In figure V.13 I have plotted the energy versus separation for three values of potential difference: 90% of V_{max} , V_{max} , and 110% of V_{max}.

FIGURE V.13



We see that for $V < V_{max}$, there are two equilibrium positions, of which the *lower* one (smaller x) is *unstable*, and we see exactly what will happen if the upper plate is displaced slightly upwards (larger x) from the unstable equilibrium position or if it is displaced slightly downwards (smaller x). The *upper* equilibrium position is stable.

If $V > V_{max}$, there is no equilibrium position, and x goes down to zero – i.e. the plates clamp together.

5.13 *Sharing a Charge Between Two Capacitors*

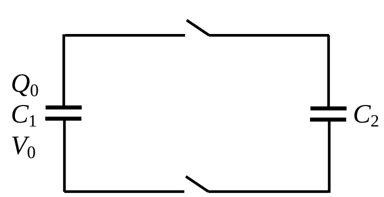


FIGURE V.14

We have two capacitors. C_2 is initially uncharged. Initially, C_1 bears a charge Q_0 and the potential difference across its plates is V_0, such that

$$Q_0 = C_1 V_0\,, \qquad 5.13.1$$

and the energy of the system is

$$E_0 = \tfrac{1}{2} C_1 V_0^2\,. \qquad 5.13.2$$

We now close the switches, so that the charge is shared between the two capacitors:

FIGURE V.15

The capacitors C_1 and C_2 now bear charges Q_1 and Q_2 such that $Q_0 = Q_1 + Q_2$ and

$$Q_1 = \frac{C_1}{C_1 + C_2} Q_0 \qquad \text{and} \qquad Q_2 = \frac{C_2}{C_1 + C_2} Q_0\,. \qquad 5.13.3\text{a,b}$$

The potential difference across the plates of either capacitor is, of course, the same, so we can call it V without a subscript, and it is easily seen, by applying $Q = CV$ to either capacitor, that

$$V = \frac{C_1}{C_1 + C_2} V_0\,. \qquad 5.13.4$$

We can now apply $E = \tfrac{1}{2} CV^2$ to each capacitor in turn to find the energy stored in each. We find for the energies stored in the two capacitors:

$$E_1 = \frac{C_1^3 V_0^2}{2(C_1 + C_2)^2} \qquad \text{and} \qquad E_2 = \frac{C_2 C_1^2 V_0^2}{2(C_1 + C_2)^2}\,. \qquad 5.13.5\text{a,b}$$

The total energy stored in the two capacitors is the sum of these, which is

$$E = \frac{C_1^2 V_0^2}{2(C_1 + C_2)}\,, \qquad 5.13.6$$

which can also be written

$$E = \frac{C_1}{C_1 + C_2} E_0 . \qquad 5.13.7$$

Surprise, surprise! The energy stored in the two capacitors is less than the energy that was originally stored in C_1. What has happened to the lost energy?

A perfectly reasonable and not incorrect answer is that it has been dissipated as heat in the connecting wires as current flowed from one capacitor to the other. However, it has been found in low temperature physics that if you immerse certain metals in liquid helium they lose *all* electrical resistance and they become *superconductive.* So, let us connect the capacitors with superconducting wires. Then there is no dissipation of energy as heat in the wires – so the question remains: where has the missing energy gone?

Well, perhaps the dielectric medium in the capacitors is heated? Again this seems like a perfectly reasonable and probably not entirely incorrect answer. However, my capacitors have a *vacuum* between the plates, and are connected by superconducting wires, so that no heat is generated either in the dielectric or in the wires. Where has that energy gone?

This will have to remain a mystery for the time being, and a topic for lunchtime conversation. In a later chapter I shall suggest another explanation.

5.14 *Mixed Dielectrics*

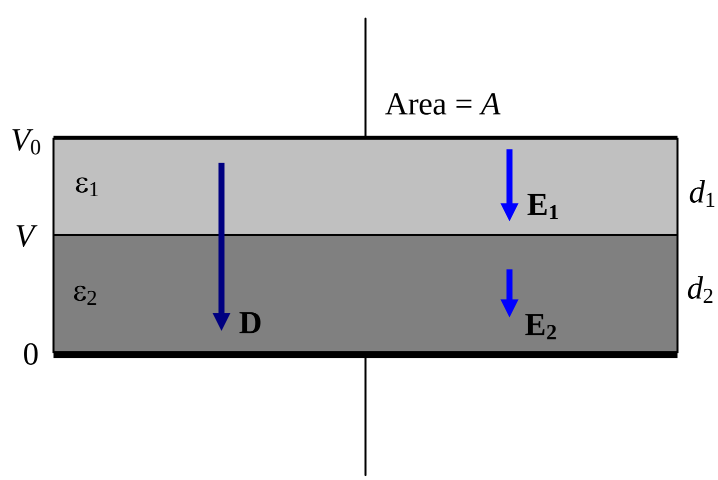
This section addresses the question: If there are two or more dielectric media between the plates of a capacitor, with different permittivities, are the electric fields in the two media different, or are they the same? The answer depends on

1. Whether by "electric field" you mean E or D;

2. The disposition of the media between the plates – i.e. whether the two dielectrics are in series or in parallel.

Let us first suppose that two media are in series (figure V.16).

FIGURE V.16



Our capacitor has two dielectrics in series, the first one of thickness d_1 and permittivity ε_1 and the second one of thickness d_2 and permittivity ε_2. As always, the thicknesses of the dielectrics are supposed to be small so that the fields within them are uniform. This is effectively two capacitors in series, of capacitances $\varepsilon_1 A/d_1$ and $\varepsilon_2 A/d_2$. The total capacitance is therefore

$$C = \frac{\varepsilon_1 \varepsilon_2 A}{\varepsilon_2 d_1 + \varepsilon_1 d_2}. \qquad 5.14.1$$

Let us imagine that the potential difference across the plates is V_0. Specifically, we'll suppose the potential of the lower plate is zero and the potential of the upper plate is V_0. The charge Q held by the capacitor (positive on one plate, negative on the other) is just given by $Q = CV_0$, and hence the surface charge density σ is CV_0/A. Gauss's law is that the total D-flux arising from a charge is equal to the charge, so that in this geometry $D = \sigma$, and this is not altered by the nature of the dielectric materials between the plates. Thus, in this capacitor, $D = CV_0/A = Q/A$ in both media. Thus D is continuous across the boundary.

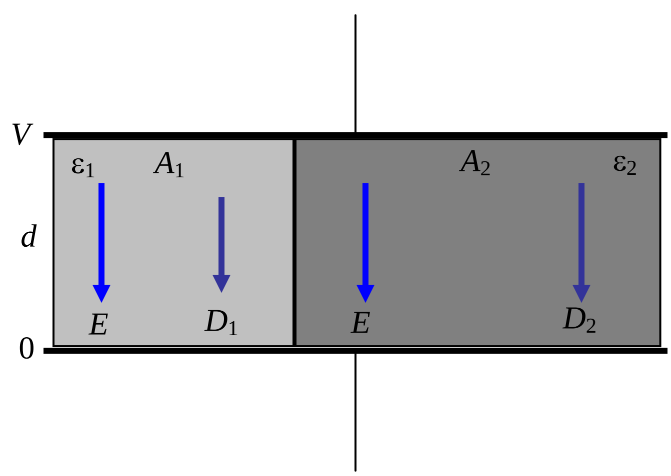
Then by application of $D = \varepsilon E$ to each of the media, we find that the E-fields in the two media are $E_1 = Q/(\varepsilon_1 A)$ and $E_2 = Q/(\varepsilon_2 A)$, the E-field (and hence the potential gradient) being larger in the medium with the smaller permittivity.

The potential V at the media boundary is given by $V/d_2 = E_2$. Combining this with our expression for E_2, and $Q = CV$ and equation 5.14.1, we find for the boundary potential:

$$V = \frac{\varepsilon_1 d_2}{\varepsilon_2 d_1 + \varepsilon_1 d_2} V_0. \qquad 5.14.2$$

Let us now suppose that two media are in parallel (figure V.17).

FIGURE V.17



This time, we have two dielectrics, each of thickness d, but one has area A_1 and permittivity ε_1 while the other has area A_2 and permittivity ε_2. This is just two capacitors in parallel, and the total capacitance is

$$C = \frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d}. \qquad 5.14.3$$

The E-field is just the potential gradient, and this is independent of any medium between the plates, so that $E = V/d$. in each of the two dielectrics. After that, we have simply that $D_1 = \varepsilon_1 E$ and $D_2 = \varepsilon_2 E$. The charge density on the plates is given by Gauss's law as $\sigma = D$, so that, if $\varepsilon_1 < \varepsilon_2$, the charge density on the left hand portion of each plate is less than on the right hand portion – although the *potential* is the same throughout each plate. (The surface of a metal is always an equipotential surface.) The two different charge densities on each plate is a result of the different *polarizations* of the two dielectrics – something that will be more readily understood a little later in this chapter when we deal with media polarization.

We have established that:

1. The component of **D** perpendicular to a boundary is continuous;
2. The component of **E** parallel to a boundary is continuous.

In figure V.18 we are looking at the D-field and at the E-field as it crosses a boundary in which $\varepsilon_1 < \varepsilon_2$. Note that D_y and E_x are the same on either side of the boundary. This results in:

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}. \qquad 5.14.4$$

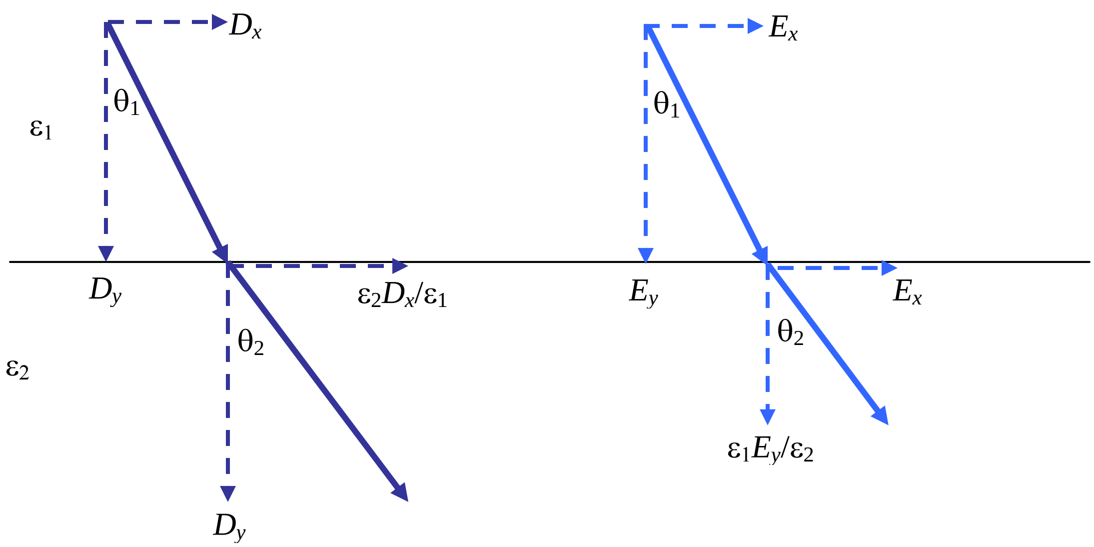


FIGURE V.18

5.15 *Changing the Distance Between the Plates of a Capacitor*

If you gradually increase the distance between the plates of a capacitor (although always keeping it sufficiently small so that the field is uniform) does the intensity of the field change or does it stay the same? If the former, does it increase or decrease?

The answers to these questions depends

1. on whether, by the field, you are referring to the E-field or the D-field;

2. on whether the plates are *isolated* or if they are *connected to the poles of a battery*.

We shall start by supposing that the plates are *isolated*.

In this case the charge on the plates is constant, and so is the charge density. Gauss's law requires that $D = \sigma$, so that D remains constant. And, since the permittivity hasn't changed, E also remains constant.

The potential difference across the plates is Ed, so, as you increase the plate separation, so the potential difference across the plates in increased. The capacitance decreases from $\epsilon A/d_1$ to $\epsilon A/d_2$ and the energy stored in the capacitor increases from $\frac{Ad_1\sigma^2}{2\epsilon}$ to $\frac{Ad_2\sigma^2}{2\epsilon}$. This energy derives from the work done in separating the plates.

Now let's suppose that the plates are *connected to a battery* of EMF V, with air or a vacuum between the plates. At first, the separation is d_1. The magnitudes of E and D are, respectively, V/d_1 and $\epsilon_0 V/d_1$. When we have increased the separation to d_2, the potential difference across the plates has not changed; it is still the EMF V of the battery. The electric field, however, is now only $E = V/d_2$ and $D = \epsilon_0 V/d_2$. But Gauss's law still dictates that $D = \sigma$, and therefore the charge density, and the total charge on the plates, is less than it was before. It has gone into the battery. In other words, in doing work by separating the plates we have recharged the battery. The energy stored in the capacitor was originally $\frac{\epsilon_0 AV^2}{2d_1}$; it is now only $\frac{\epsilon_0 AV^2}{2d_2}$. Thus the energy held in the capacitor has been reduced by $\frac{1}{2}\epsilon_0 AV^2\left(\frac{1}{d_1} - \frac{1}{d_2}\right)$.

The charge originally held by the capacitor was $\frac{\epsilon_0 AV}{d_1}$. After the plate separation has been increased to d_2 the charge held is $\frac{\epsilon_0 AV}{d_2}$. The difference, $\epsilon_0 AV\left(\frac{1}{d_1} - \frac{1}{d_2}\right)$, is the charge that has gone into the battery. The energy, or work, required to force this amount of charge into the battery against its EMF V is $\varepsilon_0 AV^2\left(\frac{1}{d_1}-\frac{1}{d_2}\right)$. Half of this came from the loss in energy held by the capacitor (see above). The other half presumably came from the mechanical work you did in separating the plates. Let's see if we can verify this.

When the plate separation is x, the force between the plates is $\frac{1}{2}QE$, which is $\frac{1}{2}\frac{\varepsilon_0 AV}{x}\cdot\frac{V}{x}$ or $\frac{\varepsilon_0 AV^2}{2x^2}\cdot$ The work required to increase x from d_1 to d_2 is $\frac{\varepsilon_0 AV^2}{2}\int_{d_1}^{d_2}\frac{dx}{x^2}$, which is indeed $\frac{1}{2}\varepsilon_0 AV^2\left(\frac{1}{d_1}-\frac{1}{d_2}\right)$. Thus this amount of mechanical work, plus an equal amount of energy from the capacitor, has gone into recharging the battery. Expressed otherwise, the work done in separating the plates equals the work required to charge the battery minus the decrease in energy stored by the capacitor.

Perhaps we have invented a battery charger (figure V.19)!

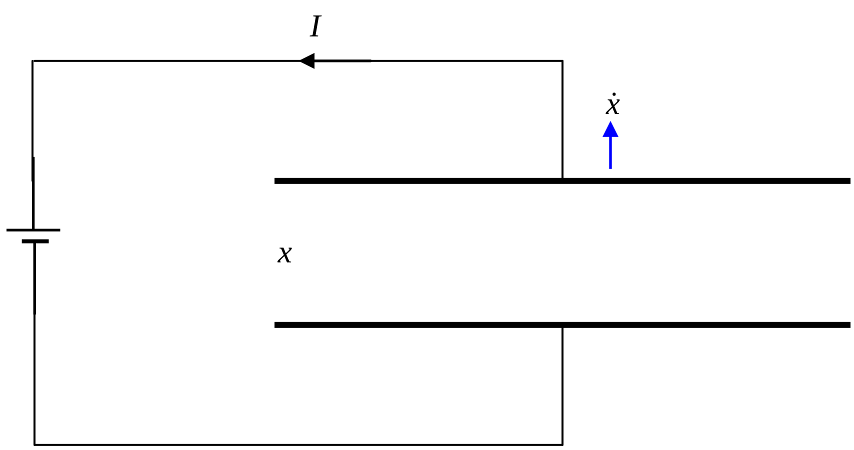


FIGURE V.19

When the plate separation is x, the charge stored in the capacitor is $Q=\frac{\varepsilon_0 AV}{x}\cdot$ If x is increased at a rate $\dot{x}$, Q will increase at a rate $\dot{Q}=-\frac{\varepsilon_0 AV\,\dot{x}}{x^2}$. That is, the capacitor will discharge (because $\dot{Q}$ is negative), and a current $I=\frac{\varepsilon_0 AV\,\dot{x}}{x^2}$ will flow counterclockwise in the circuit. (Verify that this expression is dimensionally correct for current.)

5.16 *Inserting a Dielectric into a Capacitor*

Suppose you start with two plates separated by a vacuum or by air, with a potential difference across the plates, and you then insert a dielectric material of permittivity ε_0 between the plates. Does the intensity of the field change or does it stay the same? If the former, does it increase or decrease?

The answer to these questions depends

1. on whether, by the field, you are referring to the *E*-field or the *D*-field;

2. on whether the plates are *isolated* or if they are *connected to the poles of a battery*.

We shall start by supposing that the plates are *isolated*. See figure V.20.

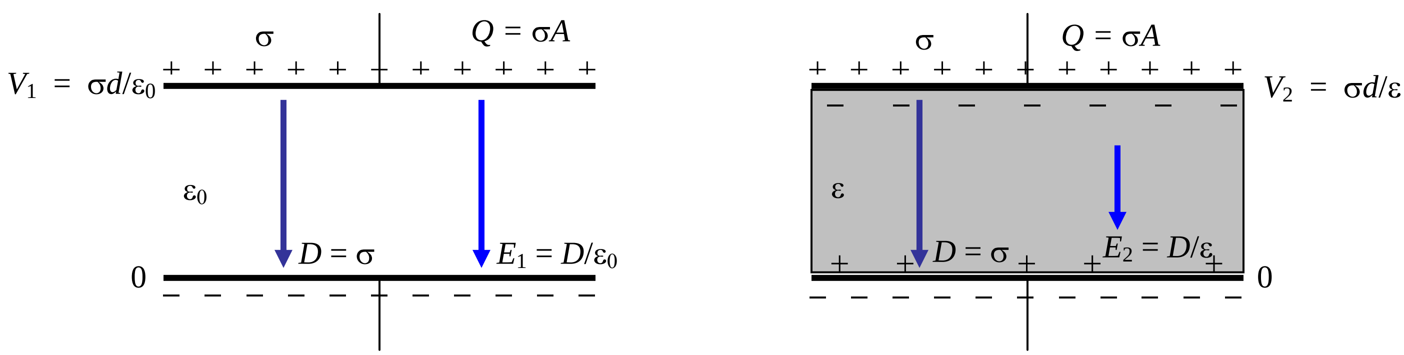


FIGURE V.20

Let Q be the charge on the plates, and σ the surface charge density. These are unaltered by the introduction of the dielectric. Gauss's law provides that $D = \sigma$, so this, too, is unaltered by the introduction of the dielectric. The electric field was, initially, $E_1 = D/\varepsilon_0$. After introduction of the dielectric, it is a little less, namely $E_1 = D/\varepsilon$.

Let us take the potential of the lower plate to be zero. Before introduction of the dielectric, the potential of the upper plate was $V_1 = \sigma d/\varepsilon_0$. After introduction of the dielectric, it is a little less, namely $V_1 = \sigma d/\varepsilon$.

Why is the electric field E less after introduction of the dielectric material? It is because the dielectric material becomes *polarized.* We saw in Section 3.6 how matter may become polarized. Either molecules with pre-existing dipole moments align themselves with the imposed electric field, or, if they have no permanent dipole moment or if they cannot rotate, a dipole moment can be induced in the individual molecules. In any case, the effect of the alignment of all these molecular dipoles is that there is a slight surplus of positive charge on the surface of the dielectric material next to the negative plate, and a slight surplus of negative charge on the surface of the dielectric material next to the positive plate. This produces an electric field opposite to the direction of the imposed field, and thus the total electric field is somewhat reduced.

Before introduction of the dielectric material, the energy stored in the capacitor was $\frac{1}{2}QV_1$. After introduction of the material, it is $\frac{1}{2}QV_2$, which is a little bit less. Thus it will require work to remove the material from between the plates. The empty capacitor will tend to suck the material in, just as the charged rod in Chapter 1 attracted an uncharged pith ball.

Now let us suppose that the plates are *connected to a battery*. (Figure V.21)

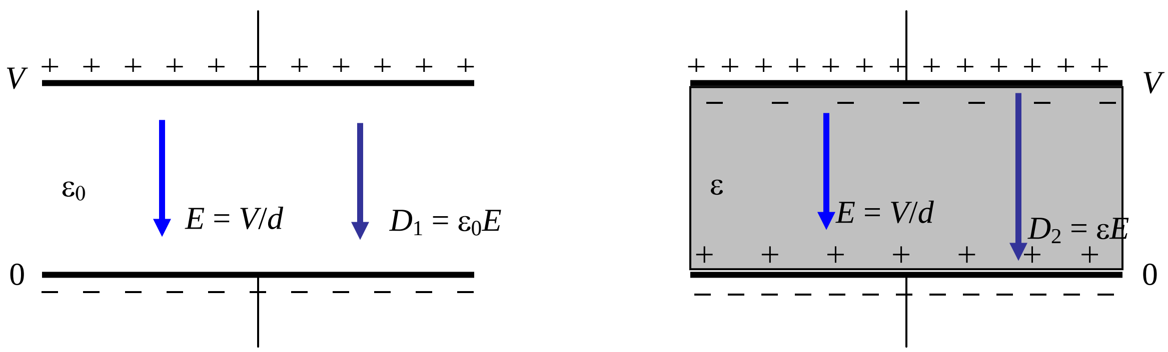


FIGURE V.21

This time the potential difference remains constant, and therefore so does the *E*-field, which is just V/d. But the *D*-field increases from $\varepsilon_0 E$ to εE, and so, therefore, does the surface charge density on the plates. This extra charge comes from the battery.

The capacitance increases from $\frac{\varepsilon_0 A}{d}$ to $\frac{\varepsilon A}{d}$ and the charge stored on the plates increases from $Q_1 = \frac{\varepsilon_0 AV}{d}$ to $Q_2 = \frac{\varepsilon AV}{d}$. The energy stored in the capacitor increases from $\frac{1}{2}Q_1V$ to $\frac{1}{2}Q_2V$.

The energy supplied by the battery = the energy dumped into the capacitor + the energy required to suck the dielectric material into the capacitor:

$$(Q_2 - Q_1)V \ = \ \tfrac{1}{2}(Q_2 - Q_1)V \ + \ \tfrac{1}{2}(Q_2 - Q_1)V.$$

You would have to do work to remove the material from the capacitor; half of the work you do would be the mechanical work performed in pulling the material out; the other half would be used in charging the battery.

In Section 5.15 I invented one type of battery charger. I am now going to make my fortune by inventing another type of battery charger.

Example 1.

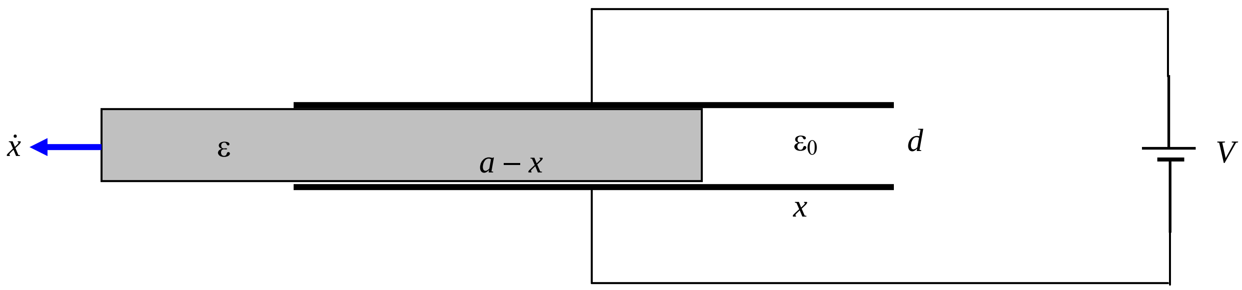


FIGURE V.22

A capacitor is formed of two square plates, each of dimensions $a \times a$, separation d, connected to a battery. There is a dielectric medium of permittivity ε between the plates. I pull the dielectric medium out at speed $\dot{x}$. Calculate the current in the circuit as the battery is recharged.

Solution.

When I have moved a distance x, the capacitance is

$$\frac{\varepsilon a(a-x)}{d} + \frac{\varepsilon_0 ax}{d} = \frac{\varepsilon a^2 - (\varepsilon - \varepsilon_0)ax}{d}.$$

The charge held by the capacitor is then

$$Q = \left[\frac{\varepsilon a^2 - (\varepsilon - \varepsilon_0)ax}{d}\right]V.$$

If the dielectric is moved out at speed $\dot{x}$, the charge held by the capacitor will increase at a rate

$$\dot{Q} = \frac{-(\varepsilon-\varepsilon_0)a\dot{x}V}{d}\,.$$

(That's negative, so Q decreases.) A current of this magnitude therefore flows clockwise around the circuit, into the battery. You should verify that the expression has the correct dimensions for current.

Example 2.

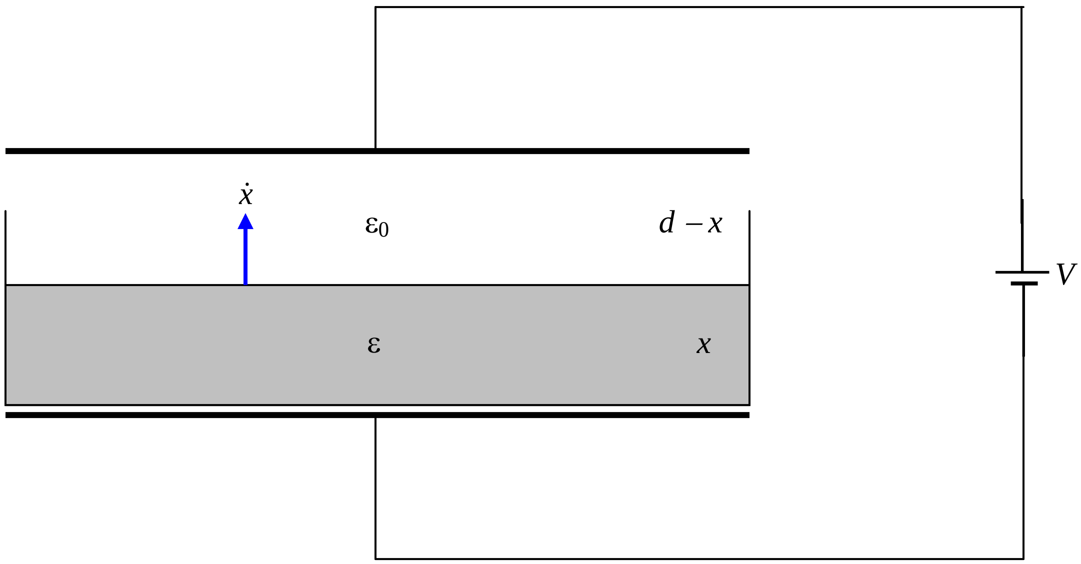


FIGURE V.23

A capacitor consists of two plates, each of area A, separated by a distance x, connected to a battery of EMF V. A cup rests on the lower plate. The cup is gradually filled with a nonconducting liquid of permittivity ε, the surface rising at a speed $\dot{x}$. Calculate the magnitude and direction of the current in the circuit.

It is easy to calculate that, when the liquid has a depth x, the capacitance of the capacitor is

$$C = \frac{\varepsilon\varepsilon_0 A}{\varepsilon d - (\varepsilon-\varepsilon_0)x}$$

and the charge held by the capacitor is then

$$Q = \frac{\varepsilon\varepsilon_0 AV}{\varepsilon d - (\varepsilon-\varepsilon_0)x}\,.$$

If x is increasing at a rate $\dot{x}$, the rate at which Q, the charge on the capacitor, is increasing is

$$\dot{Q} = \frac{\varepsilon\varepsilon_0(\varepsilon-\varepsilon_0)AV\,\dot{x}}{[\varepsilon d - (\varepsilon-\varepsilon_0)x]^2}\,.$$

A current of this magnitude therefore flows in the circuit counterclockwise, draining the battery. This current increases monotonically from zero to $\dfrac{\varepsilon(\varepsilon-\varepsilon_0)AV\,\dot{x}}{\varepsilon_0 d^2}$.

5.17 *Polarization and Susceptibility*

When an insulating material is placed in an electric field, it becomes *polarized*, either by rotation of molecules with pre-existing dipole moments or by induction of dipole moments in the individual molecules. Inside the material, D is then greater than $\varepsilon_0 E$. Indeed,

$$D = \varepsilon_0 E + P. \tag{5.17.1}$$

The excess, P, of D over $\varepsilon_0 E$ is called the *polarization* of the medium. It is dimensionally similar to, and expressed in the same units as, D; that is to say C m^{-2}. Another way of looking at the polarization of a medium is that it is the *dipole moment per unit volume.*

In vector form, the relation is

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}. \tag{5.17.2}$$

If the medium is isotropic, all three vectors are parallel.

Some media are more susceptible to becoming polarized in a polarizing field than others, and the ratio of P to $\varepsilon_0 E$ is called the electric *susceptibility* χ_e of the medium:

$$P = \chi_e\varepsilon_0 E. \tag{5.17.3}$$

This implies that P is linearly proportional to E but only if χ_e is independent of E, which is by no means always the case, but is good for small polarizations.

When we combine equations 5.17.1 and 5.17.3 with $D = \varepsilon E$ and with $\varepsilon_r = \varepsilon/\varepsilon_0$, the *relative permittivity* or *dielectric constant*, we obtain

$$\chi_e = \varepsilon_r - 1. \qquad 5.17.4$$

5.18 *Discharging a Capacitor Through a Resistor*

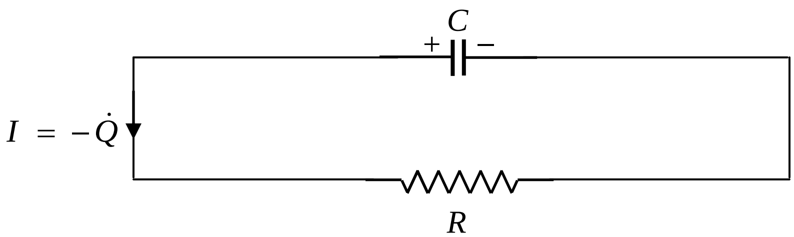


FIGURE V.24

What you have to be sure of in this section and the following section is to get the *signs* right. For example, if the charge held in the capacitor at some time is Q, then the symbol $\dot{Q}$, or dQ/dt, means the rate of increase of Q with respect to time. If the capacitor is discharging, $\dot{Q}$ is negative. Expressed otherwise, the symbol to be used for the rate at which a capacitor is *losing* charge is $-\dot{Q}$.

In figure V.24 a capacitor is discharging through a resistor, and the current as drawn is given by $I = -\dot{Q}$. The potential difference across the plates of the capacitor is Q/C, and the potential difference across the resistor is $IR = -\dot{Q}R$.

Thus:
$$\frac{Q}{C} - IR = \frac{Q}{C} + \dot{Q}R = 0. \qquad 5.18.1$$

On separating the variables (Q and t) and integrating we obtain

$$\int_{Q_0}^{Q}\frac{dQ}{Q} = -\frac{1}{RC}\int_0^t dt, \qquad 5.18.2$$

where Q_0 is the charge in the capacitor at $t = 0$.

Hence
$$Q = Q_0 e^{-t/(RC)}. \qquad 5.18.3$$

Here RC is the *time constant*. (Verify that it has the dimensions of time.) It is the time for the charge to be reduced to $1/e = 36.8\%$ of the initial charge. The half life of the charge is $RC\ln 2 = 0.6931RC$.

5.19 *Charging a Capacitor Through a Resistor*

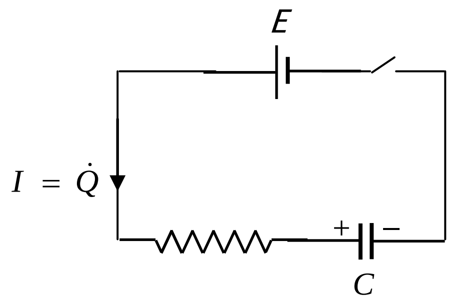


FIGURE V.25

This time, the charge on the capacitor is increasing, so the current, as drawn, is $+\dot{Q}$. Thus

$$E - \dot{Q}R - \frac{Q}{C} = 0. \qquad 5.19.1$$

Whence:
$$\int_0^Q \frac{dQ}{EC - Q} = \frac{1}{RC}\int_0^t dt. \qquad 5.19.2$$

[Note: Don't be tempted to write this as $\int_0^Q \frac{dQ}{Q - EC} = -\frac{1}{RC}\int_0^t dt$. Remember that, at any finite t, Q is less than its asymptotic value EC, and you want to keep the denominator of the left hand integral positive.]

Upon integrating, we obtain

$$Q = EC\left(1 - e^{-t/(RC)}\right). \qquad 5.19.3$$

Thus the charge on the capacitor asymptotically approaches its final value EC, reaching 63% $(1 - e^{-1})$ of the final value in time RC and half of the final value in time $RC \ln 2 = 0.6931\, RC$.

The potential difference across the plates increases at the same rate. Potential difference cannot change instantaneously in any circuit containing capacitance.

How does the current change with time? This is found by differentiating equation 5.19.3 with respect to time, to give $I = \frac{E}{R}e^{-t/(RC)}$. This suggests that the current grows instantaneously from zero to E/R as soon as the switch is closed, and then it decays exponentially, with time constant RC, to zero. Is this really possible? It is possible in principle if the inductance (see Chapter 12) of the circuit is zero. But the inductance of any closed circuit cannot be exactly zero, and the circuit, as drawn without any inductance whatever, is not achievable in any real circuit, and so, in a real circuit, there will not be an instantaneous change of current. Chapter 10 Section 10.15 will deal with the growth of current in a circuit that contains both capacitance and inductance as well as resistance.

Energy considerations

When the capacitor is fully charged, the current has dropped to zero, the potential difference across its plates is E (the EMF of the battery), and the energy stored in the capacitor (see Section 5.10) is $\frac{1}{2}CE^2 = \frac{1}{2}QE$. But the energy lost by the battery is QE. Let us hope that the remaining $\frac{1}{2}QE$ is heat generated in and dissipated by the resistor. The rate at which heat is generated by current in a resistor (see Chapter 4 Section 4.6) is I^2R. In this case, according to the previous paragraph, the current at time t is $I = \frac{E}{R}e^{-t/(RC)}$, so the total heat generated in the resistor is $\frac{E^2}{R}\int_0^\infty e^{-2t/(RC)} = \frac{1}{2}CE^2$, so all is well. The energy lost by the battery is shared equally between R and C.

Neon lamp

Here's a way of making a neon lamp flash periodically.

In figure V. $25\frac{1}{2}$ (sorry about the fraction – I slipped the figure in as an afterthought!), the thing that looks something like a happy face on the right is a discharge tube; the dot inside it indicates that it's not a complete vacuum inside, but it has a little bit of gas inside.

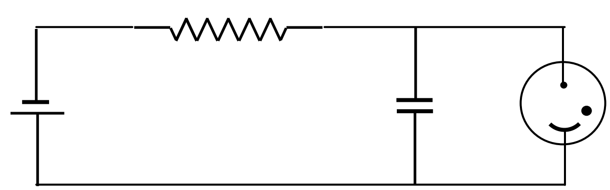
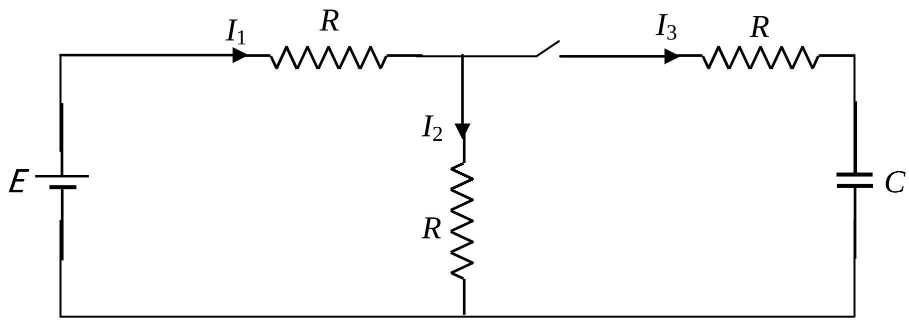


FIGURE V. $25\frac{1}{2}$

It will discharge when the potential difference across the electrodes is higher than a certain threshold. When an electric field is applied across the tube, electrons and positive ions accelerate, but are soon slowed by collisions. But, if the field is sufficiently high, the electrons and ions will have enough energy on collision to ionize the atoms they collide with, so a cascading discharge will occur. The potential difference rises exponentially on an *RC* time-scale until it reaches the threshold value, and the neon tube suddenly discharges. Then it starts all over again.

Problem

Here is a problem that will give practice in charging a capacitor, applying Kirchhoff's rules, and solving differential equations.



In the above circuit, while the switch is open, $I_1 = I_2 = \mathcal{E}/(2R)$ and $I_3 = 0$. This will also be the situation long after the switch is closed and the capacitor is charged. But we want to investigate what happens in the brief moments while the capacitor is being charged. And what will be the final charge in the capacitor?

We apply Kirchhoff's rules:

$$\mathcal{E} = I_1R + I_2R \qquad 5.19.4$$

$$I_3R + Q/C - I_2R = 0 \qquad 5.19.5$$

$$I_1 = I_2 + I_3, \qquad 5.19.6$$

Here Q is the charge on the capacitor at some time.

Eliminate I_1 and I_2 to get a single equation in I_3.

$$\mathcal{E} = \frac{2Q}{C} + 3I_3R. \qquad 5.19.7$$

But $I_3 = \frac{dQ}{dt}$, so we have differential equation in Q and the time t.

$$\frac{dQ}{dt} + \frac{2}{3RC}Q = \frac{\mathcal{E}}{3R}. \qquad 5.19.8$$

This is of the form $\frac{dy}{dx} + ay = b$, and those experienced with differential equations will have no difficulty in arriving at the solution

$$Q = \tfrac{1}{2}EC + Ae^{-\frac{2t}{3RC}} \qquad 5.19.9$$

With the initial condition that $Q = 0$ when $t = 0$, this becomes

$$\underline{Q = \tfrac{1}{2}EC\left(1 - e^{-\frac{2t}{3RC}}\right)}. \qquad 5.19.10$$

Thus the final charge in the capacitor is $\tfrac{1}{2}EC$.

The current I_3 is found by differentiating equation 5.19.10 with respect to time, and the other currents are found from Kirchhoff's rules (equations 5.19.4-6). I make them:

$$I_1 = \frac{E}{2R}\left(1 + \tfrac{1}{3}e^{-\frac{2t}{3RC}}\right) \qquad 5.19.11$$

$$I_2 = \frac{E}{2R}\left(1 - \tfrac{1}{3}e^{-\frac{2t}{3RC}}\right) \qquad 5.19.12$$

$$I_3 = \frac{E}{3R}e^{-\frac{2t}{3RC}}. \qquad 5.19.13$$

Thus I_1 goes from initially $\frac{2E}{3R}$ to finally $\frac{E}{2R}$.

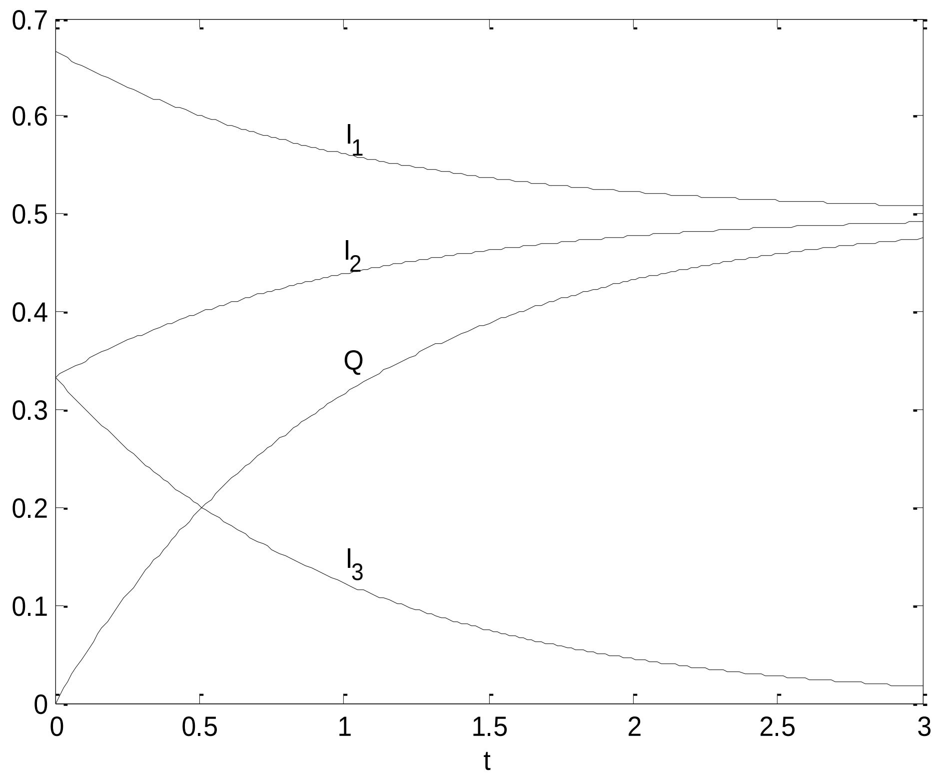
I_2 goes from initially $\frac{E}{3R}$ to finally $\frac{E}{2R}$.

I_3 goes from initially $\frac{E}{3R}$ to finally 0.

Before the switch was opened, these currents were $\frac{E}{2R}$, $\frac{E}{2R}$ and zero respectively.

Readers might wonder whether the currents can change instantaneously as soon as the switch is closed. The answer is yes, provided that the circuit has no inductance (see Chapter 10, especially Sections 10.12-15, which deal with the growth of current in a circuit that has inductance). In practice no circuit can be entirely free from inductance; apart from the inductance of any circuit components, any circuit that forms a closed loop (as all circuits must) must have a small inductance. The inductance may be very small, which means that the change of current at the instant when the switch is closed is very rapid. It is not, however, instantaneous.

Here are graphs of the currents and of Q as a function of time. Currents are expressed in units of E/R, Q in units of EC, and time in units of $\frac{3}{2}RC$.



There is a similar problem involving an inductor in Chapter 10, Section 10.12.

Integrating and differentiating circuits.

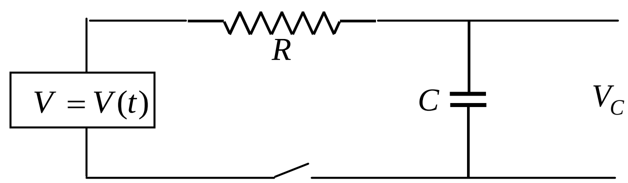
We look now at what happens if we connect a resistor and a capacitor in series across a voltage source that is varying with time, and we shall show that, provided some conditions are satisfied, the potential difference across the capacitor is the time integral of the input voltage, while the potential difference across the resistor is the time derivative of the input voltage.

We have seen that, if we connect a resistor and a capacitor in series with a battery of EMF E, the charge in the capacitor will increase according to $Q = EC\left(1 - e^{-\frac{t}{RC}}\right)$,

asymptotically approaching $Q = EC$, and reaching $1 - e^{-1} = 0.632$ of this value in time

RC. Note that, when $t \ll RC$, the current will be large, and the charge in the capacitor will be small. Most of the potential drop in the circuit will be across the resistor, and relatively little across the capacitor. After a long time, however, the current will be low, and the charge will be high, so that most of the potential drop will be across the capacitor, and relatively little across the resistor. The potential drops across R and C will be equal at a time $t = RC \ln 2 = 0.693RC$.

Suppose that, instead of connecting R and C to a battery of constant EMF, we connect it to a source whose voltage varies with time, $V(t)$. How will the charge in C vary with time?



The relevant equation is $V = IR + Q/C$, in which I, Q and V are all functions of time. Since $I = \dot{Q}$, the differential equation showing how Q varies with time is

$$\frac{dQ}{dt} + \frac{1}{RC}Q = \frac{V}{R} \qquad 5.19.14$$

The integration of this equation is made easy if we multiply both sides by $e^{\frac{t}{RC}}$. (Those who are experienced in solving differential equations will readily think of this step. Those who are less experienced might not immediately think of it, but will soon see that it is a useful step.) We then obtain

$$e^{\frac{t}{RC}}\frac{dQ}{dt} + \frac{1}{RC}e^{\frac{t}{RC}}Q = \frac{d}{dt}\left(Qe^{\frac{t}{RC}}\right) = \frac{V}{R}e^{\frac{t}{RC}} \qquad 5.19.15$$

Thus the answer to our question is

$$Q = \frac{e^{-\frac{t}{RC}}}{R}\int Ve^{\frac{t}{RC}}dt. \qquad 5.19.16$$

If $V = E$ and is independent of time, this reduces to the familiar $Q = EC\left(1 - e^{-\frac{t}{RC}}\right)$.

The potential difference across C increases, of course, as

$$V_C = \frac{e^{-\frac{t}{RC}}}{RC}\int Ve^{\frac{t}{RC}}dt. \qquad 5.19.17$$

While t is very much shorter than the time constant RC, by which I mean short enough that $e^{\frac{t}{RC}}$ is very close to 1, this becomes

$$V_C = \frac{1}{RC}\int V dt. \qquad 5.19.18$$

That is why this circuit is called an *integrating circuit*. The output voltage across C is $1/(RC)$ times the time integral of the input voltage V. This is also true if the input voltage is a periodic function of time with a period that is very much shorter than the time constant.

By way of example, suppose that $V = at^2$. If we put this in the right hand side of equation 5.19.17 and integrate, with initial condition $V_C = 0$ when $t = 0$, (do it!), we obtain

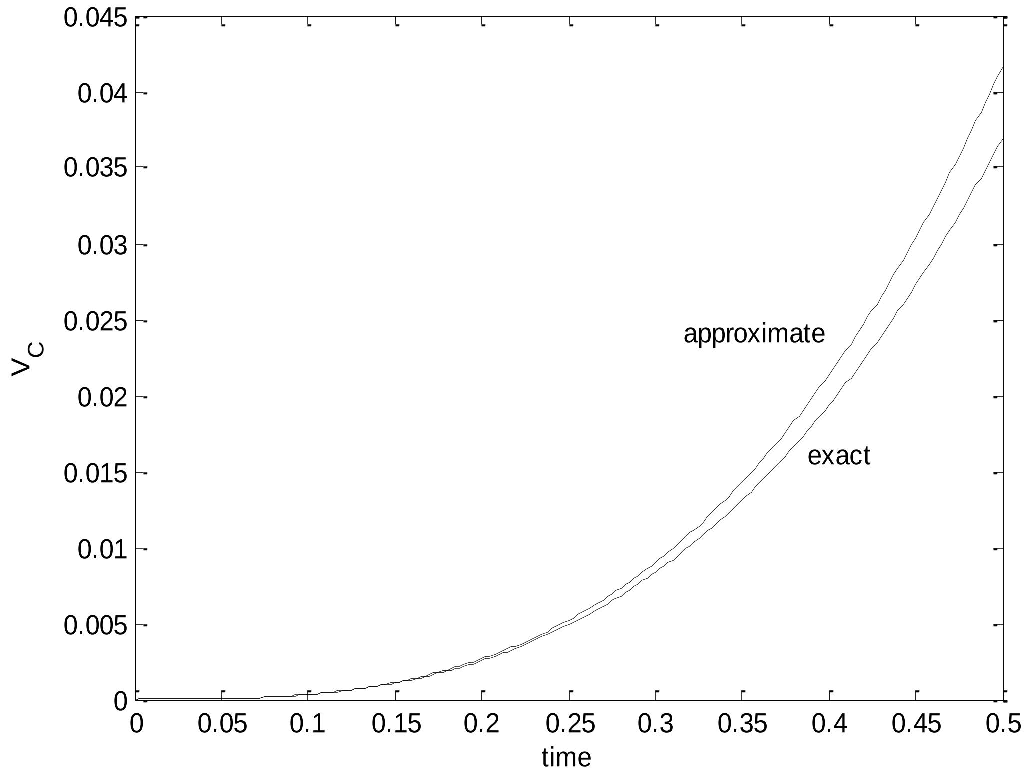
$$V_C = aR^2C^2\left(\frac{t^2}{R^2C^2} - \frac{2t}{RC} + 2 - 2e^{-\frac{t}{RC}}\right). \qquad 5.19.19$$

For example, suppose the input voltage varies as $V = 5t^2$ volts, where t is in seconds. If $R = 500\ \Omega$ and $C = 400\ \mu\text{F}$, what will be the potential differnece across the capacitor after 0.3 s? We immediately see that RC = 0.2 s and $t/(RC) = 1.5$. Substitute SI numbers in equation 5.19.19 to obtain V_C = 0.161 V.

If I write $y = \frac{V_C}{aR^2C^2}$ and $x = \frac{t}{RC}$ equation 5.19.19 in dimensionless form becomes

$$y = x^2 - 2x + 2 - 2e^{-x}. \qquad 5.19.20$$

If you Taylor expand this as far as x^3 (do it!), you get $y = \frac{1}{3}x^3$, which is just what you would get by using equation 5.19.18, the equation which is an approximation for a time that is short compared with RC. The approximation is good as long as $\left(\frac{t}{RC}\right)^4$ is negligible. I show equation 5.19.20 and $y = \frac{1}{3}x^3$ in the graph below, in which V_C is in units of aR^2C^2 and T is in units of RC.



Equation 5.19.17 (or, for short time intervals, equation 5.19.18) gives us the voltage across C as a function of time. What about the voltage across R? That is evidently

$$V_R = V - \frac{e^{-\frac{t}{RC}}}{RC}\int V e^{\frac{t}{RC}} dt. \qquad 5.19.21$$

Differentiate with respect to time:

$$\frac{dV_R}{dt} = \frac{dV}{dt} - \frac{1}{RC}\left(e^{-\frac{t}{RC}} V e^{\frac{t}{RC}} - \frac{e^{-\frac{t}{RC}}}{RC}\int V e^{\frac{t}{RC}} dt \right) \qquad 5.19.22$$

$$= \frac{dV}{dt} - \frac{1}{RC}(V - V_C) = \frac{dV}{dt} - \frac{V_R}{RC}$$

If the time constant is small so that $\frac{dV_R}{dt} << \frac{V_R}{RC}$, this becomes

$$V_R = RC\frac{dV}{dR}, \qquad 5.20.23$$

so that the voltage across *R* is *RC* times the time derivative of the input voltage *V*. Thus we have a *differentiating circuit.*

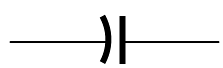
Note that, in the *integrating* circuit, the circuit must have a *large time constant* (large *R* and *C*) and time variations in *V* are *rapid* compared with *RC*. The output voltage across *C* is then $\frac{1}{RC}\int V dt$. In the *differentiating* circuit, the circuit must have a *small time constant*, and time variations in *V* are *slow* compared with *RC*. The output voltage across *R* is then $\frac{dV}{dR}$.

5.20 *Real Capacitors*

Real capacitors can vary from huge metal plates suspended in oil to the tiny cylindrical components seen inside a radio. A great deal of information about them is available on the Web and from manufacturers' catalogues, and I only make the briefest remarks here.

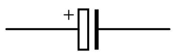
A typical inexpensive capacitor seen inside a radio is nothing much more than two strips of metal foil separated by a strip of plastic or even paper, rolled up into a cylinder much like a Swiss roll. Thus the separation of the "plates" is small, and the area of the plates is as much as can be conveniently rolled into a tiny radio component.

In most applications it doesn't matter which way round the capacitor is connected. However, with some capacitors it is intended that the outermost of the two metal strips be grounded ("earthed" in UK terminology), and the inner one is shielded by the outer one from stray electric fields. In that case the symbol used to represent the capacitor is



The curved line is the outer strip, and is the one that is intended to be grounded. It should be noted, however, that not everyone appears to be aware of this convention or adheres to it, and some people will use this symbol to denote *any* capacitor. Therefore care must be taken in reading the literature to be sure that you know what the writer intended, and, if you are describing a circuit yourself, you must make very clear the intended meaning of your symbols.

There is a type of capacitor known as an *electrolytic capacitor*. The two "plates" are strips of aluminium foil separated by a conducting paste, or electrolyte. One of the foils is covered by an extremely thin layer of aluminium oxide, which has been electrolytically deposited, and it is this layer than forms the dielectric medium, not the paste that separates the two foils. Because of the extreme thinness of the oxide layer, the capacitance is relatively high, although it may not be possible to control the actual thickness with great precision and consequently the actual value of the capacitance may not be known with great precision. It is very important that an electrolytic capacitor be corrected the right way round in a circuit, otherwise electrolysis will start to remove the oxide layer from one foil and deposit it on the other, thus greatly changing the capacitance. Also, when this happens, a current may pass through the electrolyte and heat it up so much that the capacitor may burst open with consequent danger to the eyes. The symbol used to indicate an electrolytic capacitor is:



The side indicated with the plus sign (which is often omitted from the symbol) is to be connected to the positive side of the circuit.

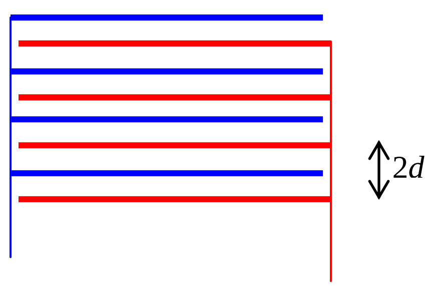
When you tune your radio, you will usually find that, as you turn the knob that changes the wavelength that you want to receive, you are changing the capacitance of a variable air-spaced capacitor just behind the knob. A variable capacitor can be represented by the symbol

Such a capacitor often consists of two sets of interleaved partiallyoverlapping plates, one set of which can be rotated with respect to the other, thus changing the overlap area and hence the capacitance.

Thinking about this suggests to me a couple of small problems for you to amuse yourself with.

Problem 1.

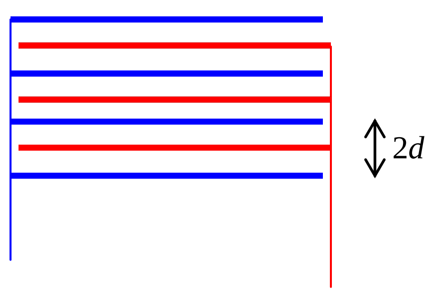
FIGURE V.26



A capacitor (figure V.26) is made from two sets of four plates. The area of each plate is *A* and the spacing between the plates in each set is 2*d*. The two sets of plates are interleaved, so that the distance between the plates of one set and the plates of the other is *d*. What is the capacitance of the system?

Problem 2

FIGURE V.27



This is just like Problem 1, except that one set has four plates and the other has three. What is the capacitance now?

Solutions. The answer to the first problem is $7\varepsilon_0 A/d$ and the answer to the second problem is $6\varepsilon_0 A/d$ – but it isn't good enough just to assert that this is the case. We must give some reasons.

Let us suppose that the potential of the left-hand (blue) plates is zero and the potential of the right-hand (blue) plates is V. The electric field in each space is V/d and $D = \varepsilon_0 V/d$. The surface charge density on each plate, by Gauss's theorem, is therefore $2\varepsilon_0 V/d$ except for the two end plates, for which the charge density is just $\varepsilon_0 V/d$. The total charge held in the capacitor of Problem 1 is therefore $\varepsilon_0 AV/d + 3 \times 2\varepsilon_0 AV/d = 7\varepsilon_0 AV/d$, and the capacitance is therefore $7\varepsilon_0 A/d$. For Problem 2, the blue set has two end-plates and two middle-plates, so the charge held is $2 \times \varepsilon_0 AV/d + 2 \times 2\,\varepsilon_0 AV/d = 6\varepsilon_0 AV/d$. The red set has three middle- plates and no end-plates, so the charge held is $3 \times 2\varepsilon_0 AV/d = 6\varepsilon_0 AV/d$. The capacitance is therefore $6\varepsilon_0 A/d$.

5.21 *More on* **E**, **D, P**, *etc.*

I'll review a few things that we have already covered before going on.

The electric field **E** between the plates of a plane parallel capacitor is equal to the potential gradient – i.e. the potential difference between the plates divided by the distance between them.

The electric field **D** between the plates of a plane parallel capacitor is equal to the surface charge density on the plates.

Suppose at first there is nothing between the plates. If you now thrust an isotropic* dielectric material of relative permittivity ε_r between the plates, what happens? Answer: If the plates are *isolated* **D** remains the same while **E** (and hence the potential difference across the plates) is reduced by a factor ε_r. If on the other hand the plates are connected to a battery, the potential difference and hence **E** remains the same while **D** (and hence the charge density on the plates) increases by a factor ε_r.

*You will have noticed the word *isotropic* here. Refer to Section 1.7 for a brief mention of an anisotropic medium, and the concept of permittivity as a tensor quantity. I'm not concerned with this aspect here.

In either case, the block of dielectric material becomes *polarized*. It develops a charge density on the surfaces that adjoin the plates. The block of material develops a *dipole moment*, and the dipole moment divided by the volume of the material – i.e. the dipole moment per unit volume – is the *polarization* **P** of the material. **P** is also equal to

$\mathbf{D} - \epsilon_0\mathbf{E}$ and, of course, to $\epsilon\mathbf{E} - \epsilon_0\mathbf{E}$. The ratio of the resulting polarization **P** to the polarizing field $\epsilon_0\mathbf{E}$ is called the electric *susceptibility* χ of the medium. It will be worth spending a few moments convincing yourself from these definitions and concepts that $\epsilon = \epsilon_0(1+\chi)$ and $\chi = \epsilon_r - 1$, where ϵ_r is the dimensionless *relative permittivity* (or *dielectric constant*) ϵ/ϵ_0.

What is happening physically inside the medium when it becomes polarized? One possibility is that the individual molecules, if they are asymmetric molecules, may already possess a *permanent dipole moment*. The molecule carbon dioxide, which, in its ground state, is linear and symmetric, O=C=O, does not have a permanent dipole moment. Symmetric molecules such as CH_4, and single atoms such as He, do not have a permanent dipole moment. The water molecule has some elements of symmetry, but it is not linear, and it does have a permanent dipole moment, of about 6 %10^{-30} C m, directed along the bisector of the HOH angle and away from the O atom. If the molecules have a permanent dipole moment and are free to rotate (as, for example, in a gas) they will tend to rotate in the direction of the applied field. (I'll discuss that phrase "tend to" in a moment.) Thus the material becomes polarized.

A molecule such as CH_4 is symmetric and has no permanent dipole moment, but, if it is placed in an external electric field, the molecule may become distorted from its perfect tetrahedral shape with neat 109º angles, because each pair of CH atoms has a dipole moment. Thus the molecule acquires an *induced dipole moment*, and the material as a whole becomes polarized. The ratio of the induced dipole moment **p** to the polarizing field **E** *polarizability* α of the molecule. Review Section 3.6 for more on this.

How about a single atom, such as Kr? Even that can acquire a dipole moment. Although there are no bonds to bend, under the influence of an electric field a preponderance of electrons will migrate to one side of the atom, and so the atom acquires a dipole moment. The same phenomenon applies, of course, to a molecule such as CH_4 in addition to the bond bending already mentioned.

Let us consider the situation of a dielectric material in which the molecules have a permanent dipole moment and are free (as in a gas, for example) to rotate. We'll suppose that, at least in a weak polarizing field, the permanent dipole moment is significantly larger than any induced dipole moment, so we'll neglect the latter. We have said that, under the influence of a polarizing field, the permanent dipole will *tend to* align themselves with the field. But they also have to contend with the constant jostling and collisions between molecules, which knock their dipole moments haywire, so they can't immediately all align exactly with the field. We might imagine that the material may become fairly strongly polarized if the temperature is fairly low, but only relatively weakly polarized at higher temperatures. Dare we even hope that we might be able to predict the variation of polarization P with temperature T? Let's have a go!

We recall (Section 3.4) that the potential energy U of a dipole, when it makes an angle θ with the electric field, is $U = -pE\cos\theta = -\mathbf{p}\bullet\mathbf{E}$. The energy of a dipole whose direction makes an angle of between θ and $\theta + d\theta$ with the field will be between U and $U + dU$, where $dU = pE \sin\theta d\theta$. What happens next requires familiarity with Boltzmann's equation for distribution of energies in a statistical mechanics. See for example my Stellar Atmospheres notes, Chapter 8, Section 8.4. The fraction of molecules having energies between U and $U + dU$ will be, following Boltzmann's equation,

$$\frac{e^{-U/(kT)}dU}{\int_{-pE}^{+pE} e^{-U/(kT)}dU}\,, \qquad 5.21.1$$

(Caution: Remember that here I'm using U for potential energy, and E for electric field.)

That is, the fraction of molecules making angles of between θ and $\theta + d\theta$ with the field is

$$\frac{pEe^{pE\cos\theta/(kT)}\sin\theta d\theta}{\int_0^\pi pEe^{pE\cos\theta/(kT)}\sin\theta d\theta} = \frac{e^{pE\cos\theta/(kT)}\sin\theta d\theta}{\int_0^\pi e^{pE\cos\theta/(kT)}\sin\theta d\theta}\,. \qquad 5.21.2$$

The component in the direction of **E** of the dipole moment of this fraction of the molecules is

$$\frac{pe^{pE\cos\theta/(kT)}\sin\theta\cos\theta d\theta}{\int_0^\pi e^{pE\cos\theta/(kT)}\sin\theta d\theta}\,, \qquad 5.21.3$$

so the component in the direction of **E** of the dipole moment all of the molecules is

$$\frac{p\int_0^\pi e^{pE\cos\theta/(kT)}\sin\theta\cos\theta d\theta}{\int_0^\pi e^{pE\cos\theta/(kT)}\sin\theta d\theta}\,, \qquad 5.21.4$$

and this expression represents the induced dipole moment in the direction of the field of the entire sample, which I'll call $p_{s.}$ The *polarization* of the sample would be this divided by its volume.

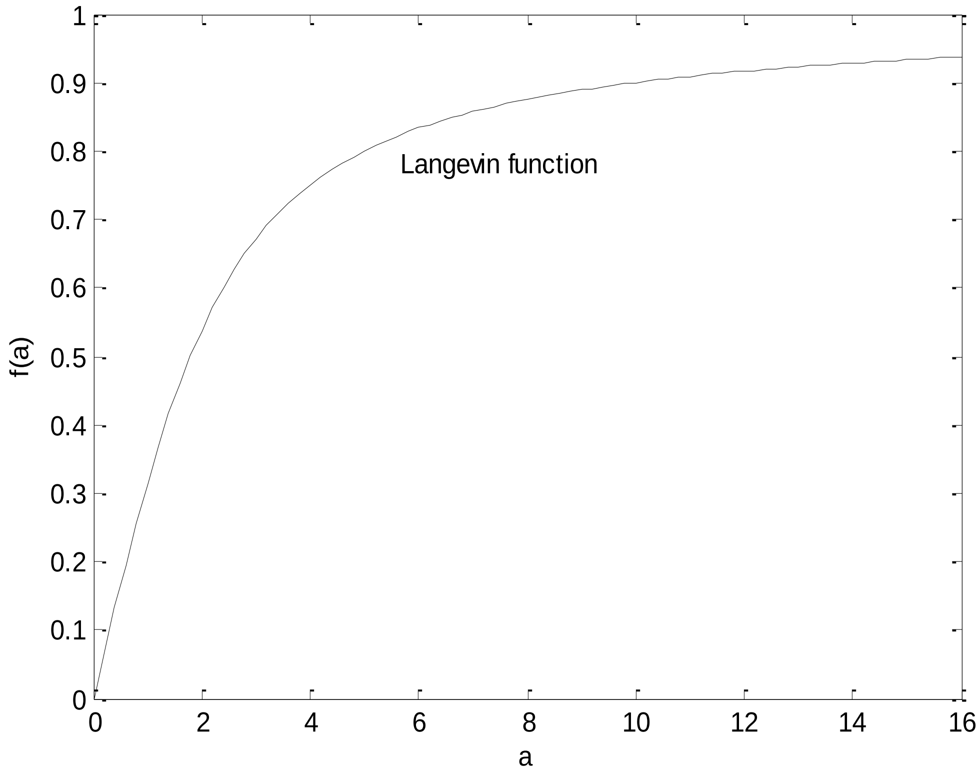
Let
$$x = \frac{pE}{kT}\cos\theta = a\cos\theta. \qquad 5.21.5$$

Then the expression for the dipole moment of the entire sample becomes (some care is needed):

$$p_s = p \times \frac{\int_{-a}^{+a} xe^x dx}{a\int_{-a}^{a} e^x dx} = p \times \left(\frac{e^a + e^{-a}}{e^a - e^{-a}} - \frac{1}{a} \right). \qquad 5.22.6$$

The expression in parentheses is called the *Langevin function*, and it was first derived in connection with the theory of paramagnetism. If your calculator or computer supports the hyperbolic coth function, it is most easily calculated as coth $a - 1/a$. If it does not support coth, calculate it as $\frac{1+b}{1-b} - \frac{1}{a}$, where $b = e^{-2a}$. In any case it is a rather interesting, even challenging, function. Let us call the expression in parentheses $f(a)$. What would the function look like it you were to plot $f(a)$ versus a? The derivative with respect to a is $\frac{1}{a^2} - \frac{4b}{(1-b)^2}$. It is easy to see that, as $a \to \infty$, the function approaches 1 and its derivative, or slope, approaches zero. But what are the function and its derivative (slope) at $a = 0$? You may find that a bit of a challenge. The answer is that, as $a \to 0$, the function approaches zero and its derivative approaches 1/3. (In fact, for small a, the Langevin function is approximately $\frac{a}{3(1-a)}$, and for very small a, it is $\frac{1}{3}a$.) Thus, for small a (i.e. hot temperatures) p_s approaches $p \times \frac{pE}{3kT}$ and no higher. The Langevin function looks like this:

FIGURE V.28

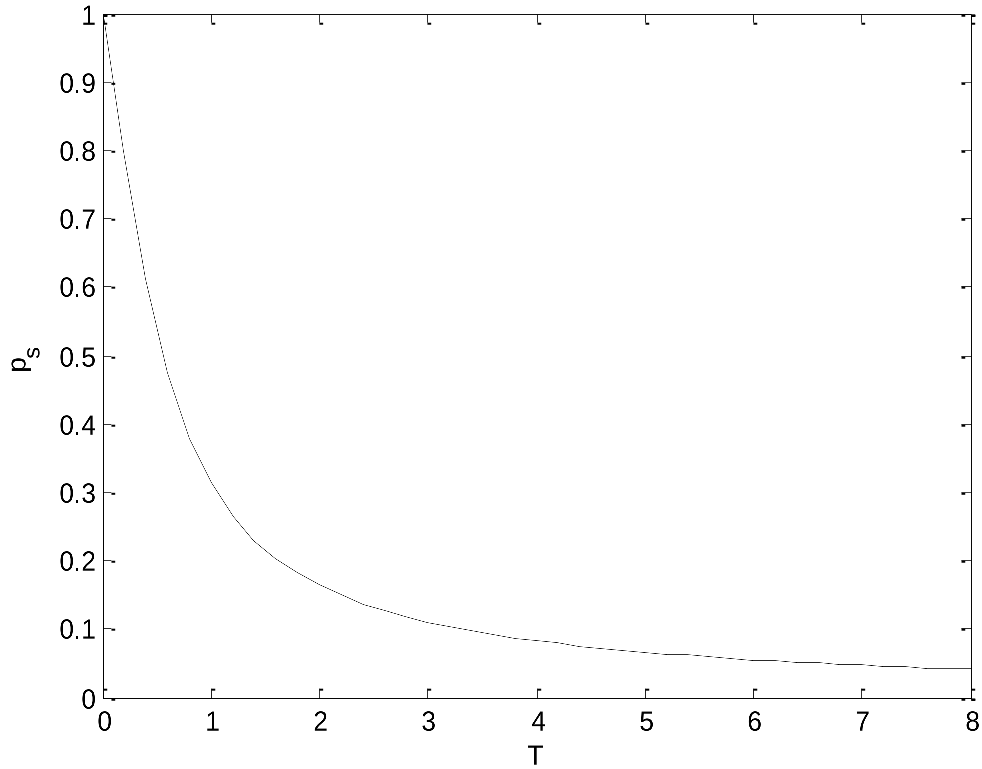


It may be more interesting to see directly how the sample dipole moment varies with temperature. If we express the sample dipole moment p_s in units of the molecular dipole moment p, and the temperature in units of pE/k, then equation 5.22.6 becomes

$$p_s = \frac{e^{1/T} + e^{-1/T}}{e^{1/T} - e^{-1/T}} - T = \coth(1/T) - T, \qquad 5.22.7$$

and that looks like this:

FIGURE V.29



The contribution to the polarization of a sample from the other two mechanisms – namely bond bending, and the pushing of electrons to one side, is independent of temperature. Thus, if we find that the polarization is temperature dependent, this tells us of the existence of a permanent dipole moment, as, for example, in methyl chloride CH_3Cl and H_2O. Indeed the temperature dependence of the polarization is part of the evidence that tells us that the water molecule is nonlinear. For small a (recall that $a = \frac{pE}{kT}$), the polarization of the material is $\frac{pE}{3kT}$, and so a graph of the polarization versus $1/T$ will be a straight line from which one can determine the dipole moent of the molecule – the greater the slope, the greater the dipole moment. One the other hand, if the polarization is temperature-independent, then the molecule is symmetric, such as methane CH_4 and OCO. Indeed the independence of the polarization on temperature is part of the evidence that tells us that CO_2 is a linear molecule.

5.22 *Dielectric material in a alternating electric field.*

We have seen that, when we put a dielectric material in an electric field, it becomes polarized, and the **D** field is now $\varepsilon\mathbf{E}$ instead of merely $\varepsilon_0\mathbf{E}$. But how long does it take to become polarized? Does it happen instantaneously? In practice there is an enormous range in relaxation times. (We may define a relaxation time as the time taken for the material to reach a certain fraction – such as, perhaps $1 - e^{-1} = 63$ percent, or whatever fraction may be convenient in a particular context – of its final polarization.) The relaxation time may be practically instantaneous, or it may be many hours.

As a consequence of the finite relaxation time, if we put a dielectric material in oscillating electric field $E = \hat{E}\cos\omega t$ (e.g. if light passes through a piece of glass), there will be a phase lag of D behind E. D will vary as $D = \hat{D}\cos(\omega t - \delta)$. Stated another way, if the E-field is $E = \hat{E}e^{i\omega t}$, the D-field will be $D = \hat{D}e^{i(\omega t - \delta)}$. Then
$\frac{D}{E} = \frac{\hat{D}}{\hat{E}}e^{-i\delta} = \varepsilon(\cos\delta - i\sin\delta)$. This can be written

$$D = \varepsilon^* E, \qquad 5.22.1$$

where $\varepsilon^* = \varepsilon' - i\varepsilon''$ and $\varepsilon' = \varepsilon\cos\delta$ and $\varepsilon'' = \varepsilon\sin\delta$.

The complex permittivity is just a way of expressing the phase difference between D and E. The magnitude, or modulus, of ε^* is ε, the ordinary permittivity in a static field.

Let us imagine that we have a dielectric material between the plates of a capacitor, and that an alternating potential difference is being applied across the plates. At some instant the charge density σ on the plates (which is equal to the D-field) is changing at a rate $\dot{\sigma}$, which is also equal to the rate of change $\dot{D}$ of the D-field), and the current in the circuit is $A\dot{D}$, where A is the area of each plate. The potential difference across the plates, on the other hand, is Ed, where d is the distance between the plates. The instantaneous rate of dissipation of energy in the material is $AdE\dot{D}$, or, let's say, the instantaneous rate of dissipation of energy per unit volume of the material is $E\dot{D}$.

Suppose $E = \hat{E}\cos\omega t$ and that $D = \hat{D}\cos(\omega t - \delta)$ so that

$$\dot{D} = -\hat{D}\omega\sin(\omega t - \delta) = -\hat{D}\omega(\sin\omega t\cos\delta - \cos\omega t\sin\delta).$$

The dissipation of energy, in unit volume, in a complete cycle (or period $2\pi/\omega$) is the integral, with respect to time, of $E\dot{D}$ from 0 to $2\pi/\omega$. That is,

$$\hat{E}\hat{D}\omega\int_0^{2\pi/\omega}\cos\omega t(\sin\omega t\cos\delta - \cos\omega t\sin\delta)dt.$$

The first integral is zero, so the dissipation of energy per unit volume per cycle is

$$\hat{E}\hat{D}\omega\sin\delta\int_0^{2\pi/\omega}\cos^2\omega t dt = \pi\hat{E}\hat{D}\omega\sin\delta.$$

Since the energy loss per cycle is proportional to sin δ, sin δ is called the *loss factor*. (Sometimes the loss factor is given as tan δ, although this is an approximation only for small loss angles.)