CHAPTER 6
MOTION IN A RESISTING MEDIUM

6.1 *Introduction*

In studying the motion of a body in a resisting medium, we assume that the resistive force on a body, and hence its deceleration, is some function of its speed. Such resistive forces are not generally conservative, and kinetic energy is usually dissipated as heat. For simple theoretical studies one can assume a simple force law, such as the resistive force is proportional to the speed, or to the square of the speed, or to some function that we can conveniently handle mathematically. For slow, laminar, nonturbulent motion through a viscous fluid, the resistance is indeed simply proportional to the speed, as can be shown at least by dimensional arguments. One thinks, for example, of Stokes's Law for the motion of a sphere through a viscous fluid. For faster motion, when laminar flow breaks up and the flow becomes turbulent, a resistive force that is proportional to the square of the speed may represent the actual physical situation better.

6.2 *Uniformly Accelerated Motion.*

Before studying motion in a resisting medium, a brief review of uniformly accelerating motion might be in order. That is, motion in which the resistance is zero. Any formulas that we develop for motion in a resisting medium must go to the formulas for uniformly accelerated motion as the resistance approaches zero.

One may imagine a situation in which a body starts with speed υ_0 and then accelerates at a rate a. One may ask three questions:

How fast is it moving after time t ?
How far has it moved in time t ?
How fast is it moving after it has covered a distance x ?

The answers to these questions are well known to any student of physics:

$$\upsilon = \upsilon_0 + at, \qquad 6.2.1$$

$$x = \upsilon_0 t + \tfrac{1}{2}at^2, \qquad 6.2.2$$

$$\upsilon^2 = \upsilon_0^2 + 2ax. \qquad 6.2.3$$

Since the acceleration is uniform, there is no need to use calculus to derive these. The first follows immediately from the meaning of acceleration. Distance travelled is the area under a speed : time graph. Figure VI.1 shows a speed : time graph for constant acceleration, and equation 6.2.2 is obvious from a glance at the graph. Equation 6.2.3 can be obtained by elimination of t between equations 6.2.1 and 6.2.2. (It can also be deduced from energy considerations, though that is rather putting the cart before the horse.)

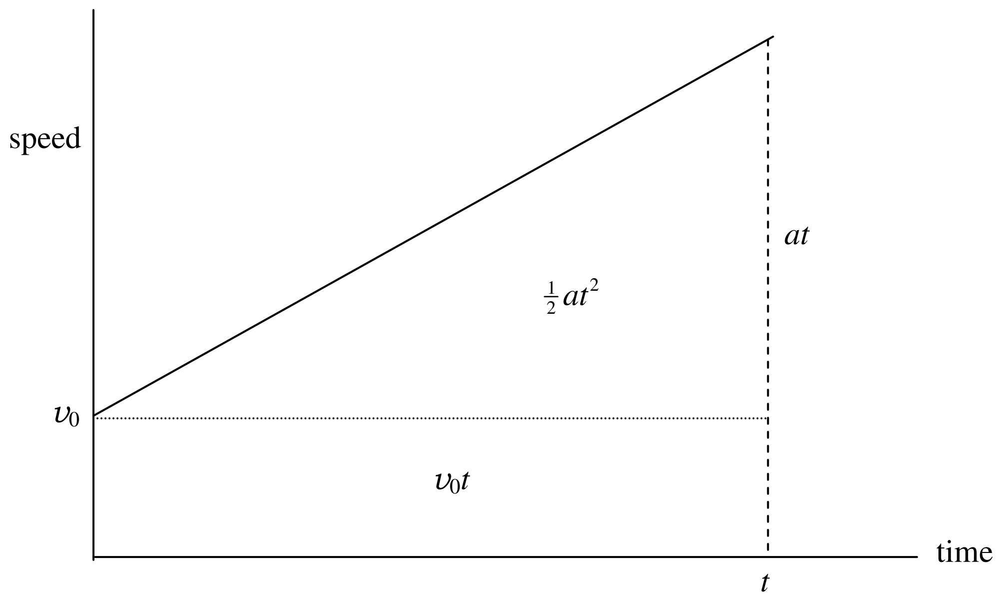


FIGURE IV.1

Nevertheless, although calculus is not necessary, it is instructive to see how calculus can be used to analyse uniformly accelerated motion, since calculus will be necessary in less simple situations. We shall be using calculus to answer the three questions posed earlier in the section.

For uniformly accelerated motion, the *equation of motion* is

$$\ddot{x} = a. \tag{6.2.4}$$

To answer the first question, we write $\ddot{x}$ as $d\upsilon/dt$, and then the integral (with initial condition x=0 when t=0) is

$$\upsilon = \upsilon_0 + at. \tag{6.2.5}$$

This is the *first time integral.*

Next, we write υ as dx/dt and integrate again with respect to time, to get

$$x = \upsilon_0 t + \tfrac{1}{2}at^2. \tag{6.2.6}$$

This is the *second time integral.*

To obtain the answer to the third question, which will be called the *space integral*, we must remember to write $\ddot{x}$ as $\upsilon\, d\upsilon/dx$. Thus the equation of motion (equation 6.2.4) is

$$\upsilon \frac{d\upsilon}{dx} = a. \tag{6.2.7}$$

When this is integrated with respect to x (with initial condition $\upsilon = \upsilon_0$ when $x = 0$) we obtain

$$\upsilon^2 = \upsilon_0^2 + 2ax. \qquad 6.2.8$$

This is the *space integral.*

Examples.

Here are a few quick examples of problems in uniformly accelerated motion. It is probably a good idea to *work in algebra* and obtain *algebraic* solutions to each problem. That is, even if you are told that the initial speed is 15 m s^{-1}, call it υ_0, or, if you are told that the height is 900 feet, call it h. You will probably find it helpful to sketch graphs either of distance versus time or speed versus time in most of the problems. One last little hint: Remember that the two solutions of a quadratic equation are equal if $b^2 = 4ac$.

1. A body is dropped from rest. The last third of the distance before it hits the ground is covered in time T. Show that the time taken for the entire fall to the ground is $5.45T$.

2. The Lady is 8 metres from the bus stop, when the Bus, starting from rest at the bus stop, starts to move off with an acceleration of 0.4 m s^{-2}. What is the least speed at which the Lady must run in order to catch the Bus?

3. A parachutist is descending at a constant speed of 10 feet per second. When she is at a height of 900 feet, her friend, directly below her, throws an apple up to her. What is the least speed at which he must throw the apple in order for it to reach her? How long does it take to reach her, what height is she at then, and what is the relative speed of parachutist and apple? Assume g = 32 ft s^{-2}. Neglect air resistance for the apple (but not for the parachutist!)

4. A lunar explorer performs the following experiment on the Moon in order to determine the gravitational acceleration g there. He tosses a lunar rock upwards at an initial speed of 15 m s^{-1}. Eight seconds later he tosses another rock upwards at an initial speed of 10 m s^{-1}. He observes that the rocks collide 16.32 seconds after the launch of the first rock. Calculate g and also the height of the collision.

5. Mr A and Mr B are discussing the merits of their cars. Mr A can go from 0 to 50 mph in ten seconds, and Mr B can go from 0 to 60 mph in 20 seconds. Mr B gives Mr A a start of one second. Assuming that each driver first accelerates uniformly to his maximum speed and thereafter each travels at uniform speed, how long does it take Mr B to catch Mr A, and how far have the cars travelled by then?

Answers

I make the answers as follows. Let me know (`jtatum at uvic dot ca`) if you think I have got any of them wrong.

2. 2.53 m s^{-1}. **3**. 230 ft s^{-1}, 7.5 s, 825 ft, 0 ft s^{-1}. **4**. 1.64 m s^{-2}, 26.4 m
5. 41 s, half a mile.

6.3 *Motion in which the Resistance is Proportional to the Speed.*

If the only force on a body is a resistive force that is proportional to its speed, the equation of motion is

$$m\ddot{x} = -b\dot{x}. \qquad 6.3.1$$

One thinks, for example, of Stokes's equation for the laminar motion of a sphere through a viscous fluid, in which the resistive force is $6\pi\eta a \upsilon$, where η is the coefficient of dynamic viscosity. If we divide both sides of the equation by the mass *m*, we obtain

$$\ddot{x} = -\gamma\dot{x}, \qquad 6.3.2$$

where $\gamma = b/m$ is the *damping constant*. It has dimension T^{-1} and SI units s^{-1}.

6.3a. *Resistive force only*

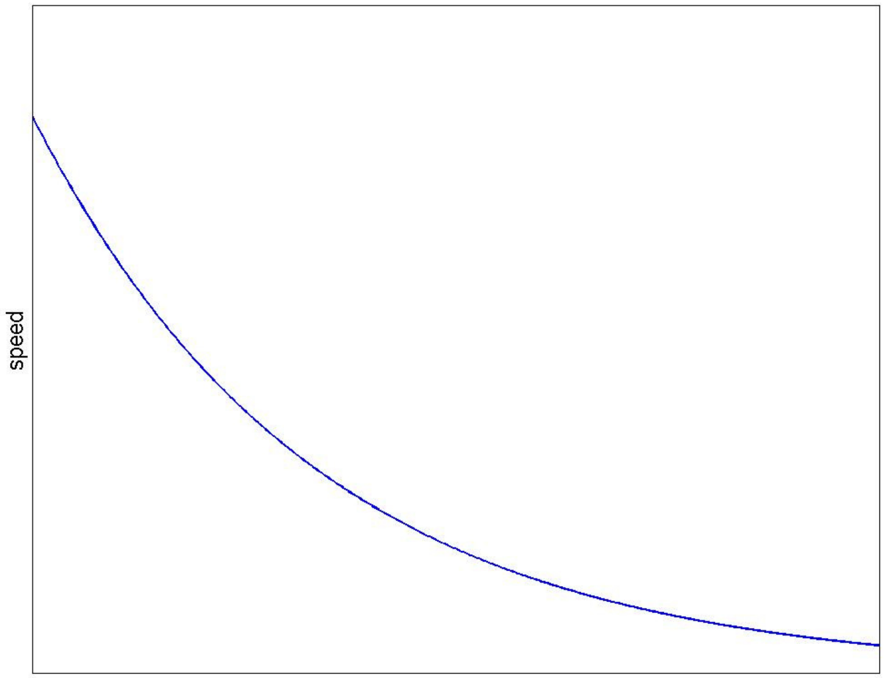
It is difficult to imagine a real situation in which the one and only force is a resistive force proportional to the speed. A body falling through the air won't do, because, in addition to the resistive force, there is the acceleration due to gravity. Perhaps we could imagine a puck sliding across the ice. The ice would have to be presumed to be completely frictionless, and the only force on the puck would be the resistance of the air. It is a slightly artificial situation, because we want the puck to be going so fast that the frictional force is negligible compared with the air resistance, but not so fast that the airflow is turbulent - but we need to start somewhere. The frictional force is, at least to a very good approximation, not a function of speed, but is constant, and we shall start by assuming that it is negligible and that the only horizontal force on the puck is air resistance and that the air resistance is proportional to the speed.

In this case, the equation of motion is indeed equation 6.3.2. To obtain the *first time integral*, we write $\dot{x}$ as υ and the first time integral is readily found to be

$$\upsilon = \upsilon_0 e^{-\gamma t}. \qquad 6.3.3$$

Here υ_0 is the initial speed. This is illustrated in figure VI.2

FIGURE VI.2



The speed is reduced to half of the initial speed in a time

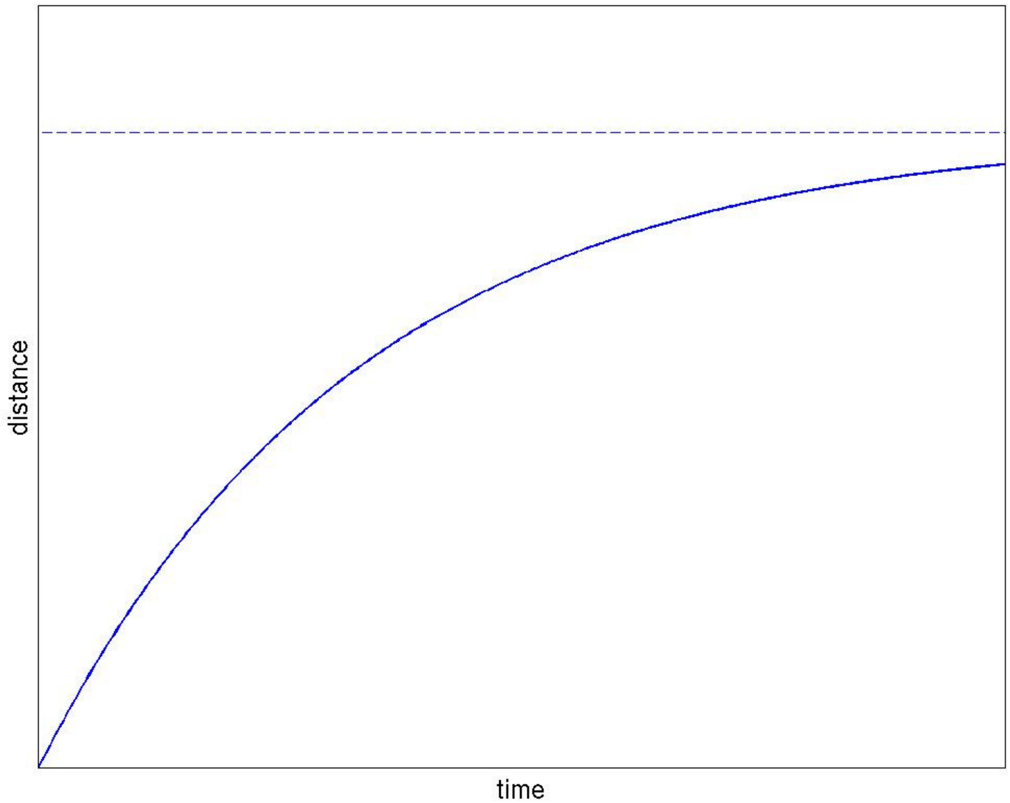
$$t_{\frac{1}{2}} = \frac{\ln 2}{\gamma} = \frac{0.693}{\gamma}\cdot \qquad 6.3.4$$

The *second time integral* is found by writing υ in equation 6.3.3 as *dx*/*dt*. Integration, with initial condition $x = 0$ when $t = 0$, gives

$$x = x_\infty\left(1 - e^{-\gamma t}\right), \qquad 6.3.5$$

where $x_\infty = \upsilon_0/\gamma$. This is illustrated in figure VI.3. It is seen that the puck travels an eventual distance of x_∞, but only after an infinite time.

FIGURE VI.3



We can obtain the *space integral* either by eliminating *t* from between the two time integrals, or by writing the equation of motion as

$$v\frac{dv}{dx} = -\gamma v. \qquad 6.3.6$$

With initial condition $v = v_0$ when $x = 0$, this becomes

$$v = v_0 - \gamma x, \qquad 6.3.7$$

which is illustrated in figure VI.4. The speed drops linearly with distance (but exponentially with time) reaching zero after having travelled a finite distance $x_\infty = v_0 / \gamma$ in an infinite time.

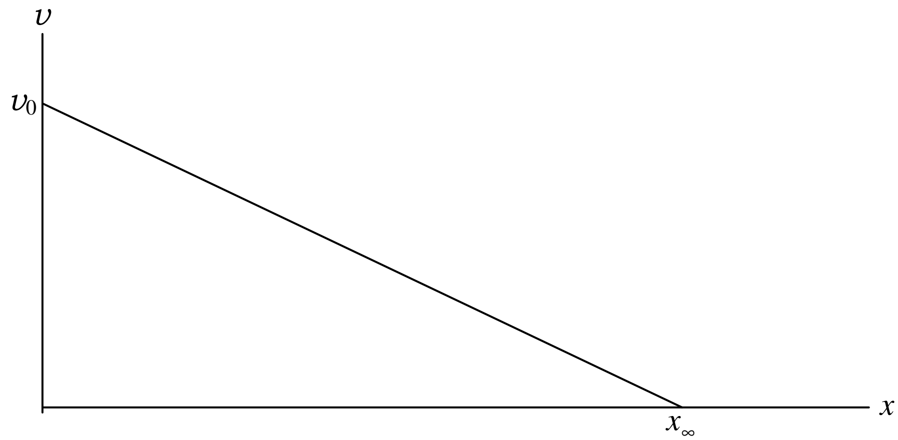


FIGURE V1.4

This analysis has assumed that the only force was the resistive force proportional to the speed. In the case of our imaginary ice puck, we were assuming that the resistive force was that of the air, the friction being negligible. Of course, as the puck slows down and the resistive force becomes less, there will come a point when the frictional force is no longer negligible compared with the ever-decreasing air resistance, so that the above equations no longer accurately describe the motion. We shall come back to this point in subsection 3c.

6.3b. *Body falling under gravity in a resisting medium, resistive force proportional to the speed.*

We are here probably considering a small sphere falling slowly through a viscous liquid, with laminar flow around the sphere, rather than a skydiver hurtling through the air. In the latter case, the airflow is likely to be highly turbulent and the resistance proportional to a higher power of the speed than the first.

We'll use the symbol y for the distance fallen. That is to say, we measure y downwards from the starting point. The equation of motion is

$$\ddot{y} = g - \gamma \upsilon, \qquad 6.3.8$$

where g is the gravitational acceleration.

The body reaches a constant speed when $\ddot{y}$ becomes zero. This occurs at a speed $\hat{\upsilon} = g/\gamma$, which is called the *terminal speed*.

To obtain the *first time integral*, we write the equation of motion as

$$\frac{dv}{dt} = \gamma(\hat{v} - v) \qquad 6.3.9$$

or

$$\frac{dv}{\hat{v} - v} = \gamma dt. \qquad 6.3.10$$

CAUTION. At this point it might be tempting to write

$$\frac{dv}{v - \hat{v}} = -\gamma dt. \qquad 6.3.11$$

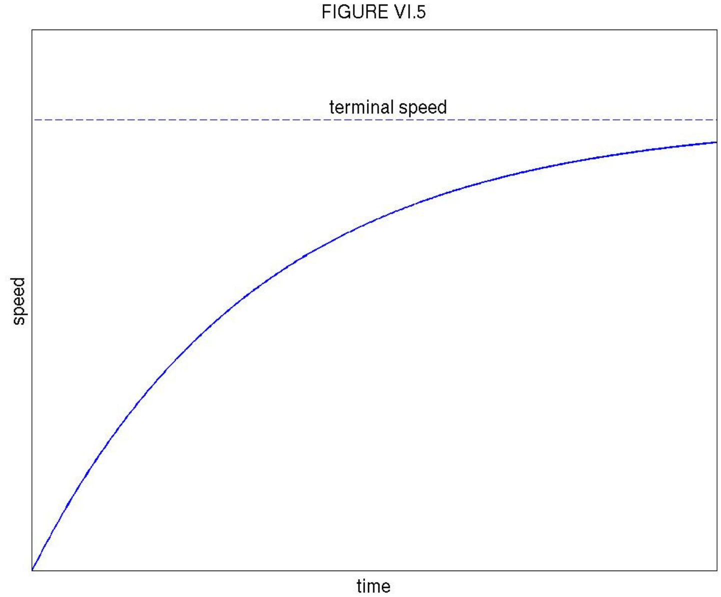
DON'T! In the middle of an exam, while covering this derivation that you know so well, you can suddenly find yourself in inextricable difficulties. The thing to note is this. If you look at the left hand side of the equation, you will anticipate that a logarithm will appear when you integrate it. Keep the denominator positive! Some mathematicians may know the meaning of the logarithm of a negative number, but most of us ordinary mortals do not - so keep the denominator positive!

With initial condition $v = 0$ when $t = 0$, the *first time integral* becomes

$$v = \hat{v}\left(1 - e^{-\gamma t}\right). \qquad 6.3.12$$

This is illustrated in figure VI.5.

FIGURE VI.5



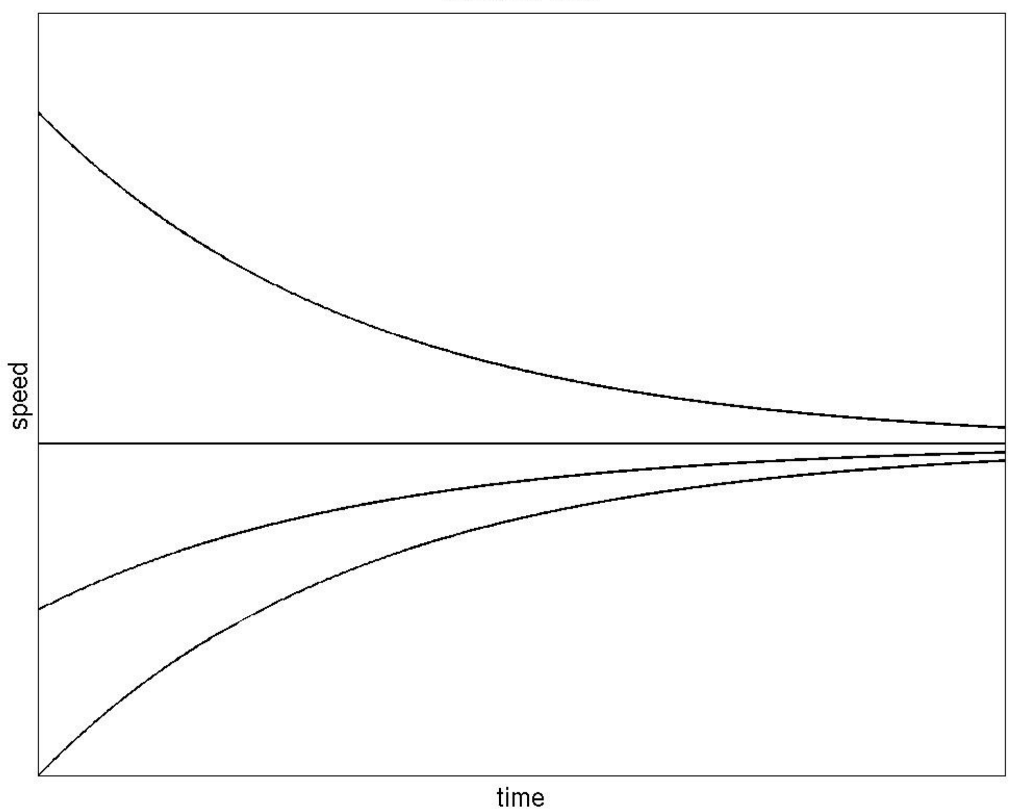
Students will have seen equations similar to this before in other branches of physics - e.g. growth of charge in a capacitor or growth of current in an inductor. That is why learning physics becomes easier all the time, because you have seen it all before in quite different contexts. Perhaps you have already noticed that third-year physics is easier than second-year physics; just think how much easier fourth-year is going to be! At any rate, υ approaches the terminal speed asymptotically, never quite reaching it, but reaching half of the terminal speed in time $(\ln 2)/\gamma = 0.693/\gamma$ (you have seen that before while studying radioactive decay), and reaching $(1 - e^{-1}) =$ 63% of the terminal speed in time $1/\gamma$.

If the body is thrown downwards, so that its initial speed is not zero but is $\upsilon = \upsilon_0$ when $t = 0$, you will write the equation of motion either as equation 6.3.10 or as equation 6.3.11, depending on whether the initial speed is slower than or faster than the terminal speed, thus ensuring that the denominator is kept firmly positive. In either case, the result is

$$\upsilon = \hat{\upsilon} + (\upsilon_0 - \hat{\upsilon})e^{-\gamma t} \qquad 6.3.13$$

Figure VI.6 shows υ as a function of t for initial conditions $\upsilon_0 = 0, \frac{1}{2}\hat{\upsilon}, \hat{\upsilon}, 2\hat{\upsilon}$.

FIGURE VI.6

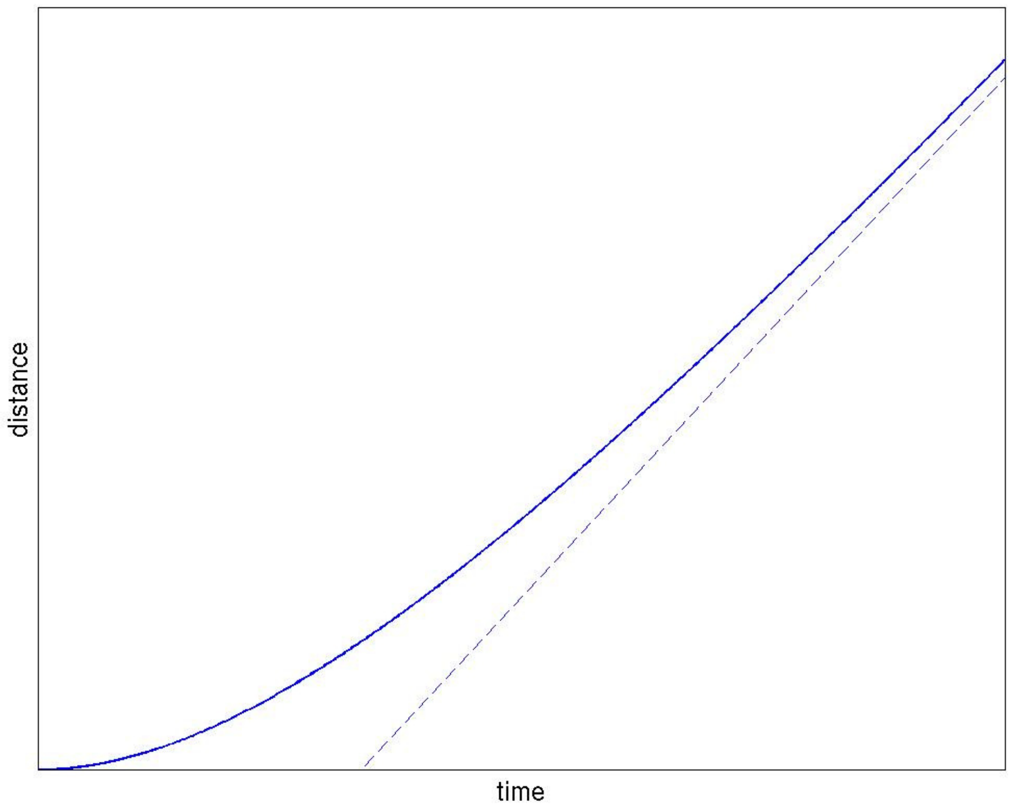


Returning to the initial condition $\upsilon = 0$ when $t = 0$, we readily find the *second time integral* to be

$$y = \hat{\upsilon}t - \frac{\hat{\upsilon}}{\gamma}\left(1 - e^{-\gamma t}\right). \qquad 6.3.14$$

You should check whether this equation is what is expected for when $t = 0$ and when t approaches infinity. The second time integral is shown in figure VI.7.

FIGURE VI.7



The *space integral* is found either by eliminating t between the first and second time integrals, or by writing $\ddot{y}$ as $\upsilon d\upsilon/dy$ in the equation of motion:

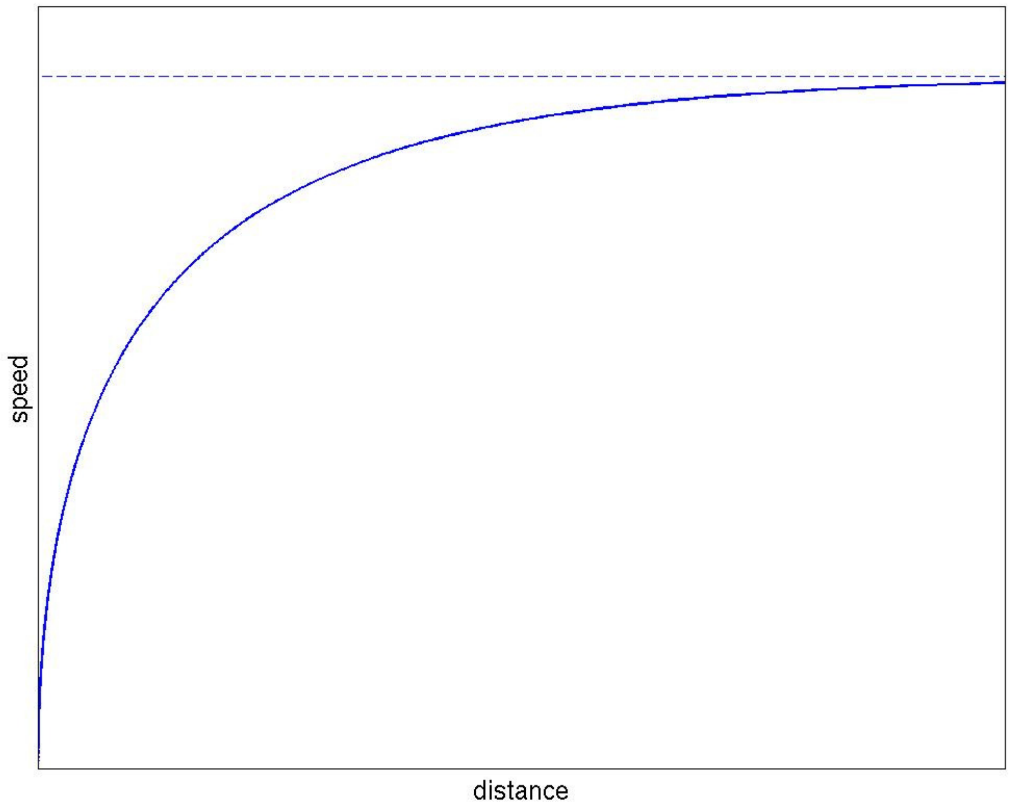
$$\upsilon\frac{d\upsilon}{dy} = \gamma(\hat{\upsilon} - \upsilon), \qquad 6.3.15$$

whence

$$y = -\frac{\hat{\upsilon}}{\gamma}\ln\left(1 - \frac{\upsilon}{\hat{\upsilon}}\right) - \frac{\upsilon}{\gamma}. \qquad 6.3.16$$

This is illustrated in figure VI.8. Notice that the equation gives y as a function of υ, but only numerical calculation will give υ for a given y.

FIGURE VI.8



Problem: Assume g = 9.8 m s^{-2} . A particle, starting from rest, is dropped through a medium such that the terminal speed is 9.8 m s^{-1}. How long will it take to fall through 9.8 m?

We are asked for t, given y, and we know the equation relating t and y - it is the *second time integral*, equation 6.3.14 - so what could be easier? We have $\gamma = g/\hat{\upsilon} = 1\,\text{s}^{-1}$, so equation 6.3.14 becomes

$$9.8 = 9.8t - 9.8\left(1 - e^{-t}\right) \qquad 6.3.17$$

and suddenly we find that it is not as easy as expected!

The equation can be written

$$f(t) = t + e^{-t} - 2 = 0. \qquad 6.3.18$$

For Newton-Raphson iteration we need

$$f'(t) = 1 - e^{-t}. \qquad 6.3.19$$

and, after some rearrangement, the Newton-Raphson iteration $(t \rightarrow t - f/f')$ becomes

$$t = \frac{1-t}{e^t - 1} + 2. \qquad 6.3.20$$

(It may be noticed that 6.3.20, which derives from the Newton-Raphson process, is merely a rearrangement of equation 6.3.18.)

Starting with an exceedingly stupid first guess of t = 100 s, the iterations proceed as follows:

t = 100.000 000 000
2.000 000 000
1.843 482 357
1.841 406 066
1.841 405 661
1.841 405 660 s

Problem: Assume g = 9.8 m s^{-2}. A particle, starting from rest, falls through a resisting medium, the damping constant being γ = 1.96 s^{-1} (i.e. $\hat{\upsilon}$ = 5 m s^{-1}). How fast is it moving after it has fallen 0.3 m?

We are asked for υ, given y. We want the *space integral*, equation 6.3.16. On substituting the data, we obtain

$$f(\upsilon) = 5\ln(1 - 0.2\upsilon) + \upsilon + 0.588 = 0. \qquad 6.3.21$$

From this,

$$f'(\upsilon) = \upsilon/(\upsilon - 5) \qquad 6.3.22$$

The Newton-Raphson process $(t \rightarrow t - f/f')$, after some algebra, arrives at

$$\upsilon = \frac{u(5\ln(0.2u) + 0.588)}{\upsilon} + 5 = \frac{u(5\ln u - 7.459189560)}{\upsilon} + 5, \qquad 6.3.23$$

where $u = 5 - \upsilon$.

This time Newton-Raphson does not allow us the luxury of an exceedingly stupid first guess, but we know that the answer must lie between 0 and 5 m s^{-1} , so our moderately intelligent first guess can be υ = 2.5 m s^{-1} .

Newton-Raphson iterations:

$$\begin{aligned} \upsilon = {} & 2.500\ 000\ 000 \\ & 2.122\ 264\ 100 \\ & 2.051\ 880\ 531 \\ & 2.049\ 766\ 247 \\ & 2.049\ 764\ 400 \text{ m s}^{-1} \end{aligned}$$

Problems

Here are four problems concerning a body falling from rest such that the resistance is proportional to the speed. Assume that $g = 9.8 \text{ m s}^{-2}$. Answers to questions 1 - 3 are to be given to a precision of 0.0001 seconds.

1. A particle falls from rest in a medium such that the damping constant is $\gamma = 1.0 \text{ s}^{-1}$. How long will it take to fall through 10 m?

2. It takes t seconds to fall through y metres. Construct a table showing t for 201 values of y going from 0 to 20 metres in steps of 0.1 metre, assuming that $\gamma = 1.0 \text{ s}^{-1}$.

3. Construct a table showing t for 201 values of y going from 0 to 20 metres in steps of 0.1 metres for γ = 0.0, 0.5, 1.0, 1,5, 2.0 s . The table is to have six columns. The first column gives the distance fallen to a precision of 0.1 metres. The remaining five columns will give the time, to a precision of 0.0001 seconds, that the body takes to fall a given distance, to a precision of 0.0001 seconds

4. Draw, by computer, a graph showing t (the dependent variable, plotted vertically) versus y (plotted horizontally) for the five values of γ in question 3.

These four problems are in order of increasing difficulty. The first is merely an exercise in solving an implicit equation (equation 6.3.14) numerically, and might serve as an introductory example of how, for example, to solve an equation by Newton-Raphson iteration (I make the answer 1.8656 s.) The last two, if started from scratch, could well take up an entire afternoon before it is solved to one's complete satisfaction. It might be observed that the graphs of question 4 could be drawn fairly easily by calculating y explicitly as a function of t, thus obviating the necessity of Newton-Raphson iteration. No such short cuts, however, can be made for constructing the table of question 3.

In fact I solved questions 3 and 4 in just a few minutes – but I did not start from scratch. As you progress through your scientific career, you will become aware that there are certain operations that you encounter time and time again. To do questions 3 and 4, for example, you need to be able to solve an equation by Newton-Raphson iteration; you need to be able to construct a table of a function $y = f\ (x\ ;\ a)$, or in this case $t = f\ (y\ ;\ \gamma)$; and you need to be able to instruct a computer to draw graphs of tabulated values. I learned long ago that all of these are problems that crop up frequently, and I therefore long ago wrote short programs (only a few lines of Fortran each) for doing each of them. All I had to do on this occasion was to marry these existing programs together, tailored to the particular functions needed. Likewise a student will recognize similar problems for which he or she frequently needs a solution. You should accumulate and keep a set of these small programs for use in the future whenever you may need them. For example, this is by no means the last time you will need Newton-Raphson iteration to solve an equation. Write a Newton-Raphson program now and keep it for future occasions!

6.3c. *Body thrown vertically upwards, initial speed υ_0*

If we measure y upwards from the ground, the equation of motion is

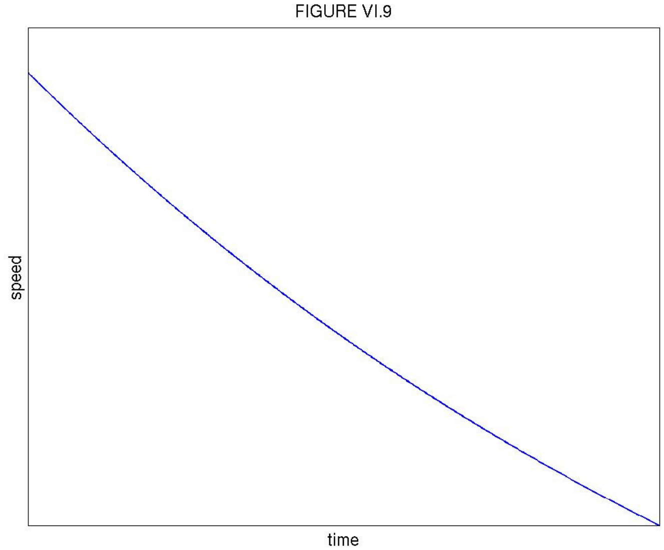
$$\ddot{y} = -g - \gamma\upsilon = -\gamma(\hat{\upsilon} + \upsilon). \qquad 6.3.24$$

The *first time integral* is

$$\upsilon = -\hat{\upsilon} + (\upsilon_0 + \hat{\upsilon})e^{-\gamma t} \qquad 6.3.25$$

and this is shown in figure VI.9.

FIGURE VI.9



It reaches a maximum height after time T, when $\upsilon = 0$ (at which time the acceleration is just $-g$):

$$T = \frac{1}{\gamma}\ln\left(1+\frac{\upsilon_0}{\hat{\upsilon}}\right). \qquad 6.3.26$$

The *second time integral* (obtained by writing υ as *dy/dt* in equation 6.3.25) and the *space integral* (obtained by writing $\ddot{y}$ as $\upsilon\, d\upsilon/dy$ in the equation of motion) require some patience, but the results are

$$y = \frac{(\upsilon_0 + \hat{\upsilon})}{\gamma}\left(1 - e^{-\gamma t}\right) - \hat{\upsilon}t, \qquad 6.3.27$$

$$\upsilon = \upsilon_0 - \gamma\, y - \hat{\upsilon}\ln\left(\frac{\hat{\upsilon}+\upsilon_0}{\hat{\upsilon}+\upsilon}\right). \qquad 6.3.28$$

These are illustrated in figures VI.10 and VI.11.

FIGURE VI.10

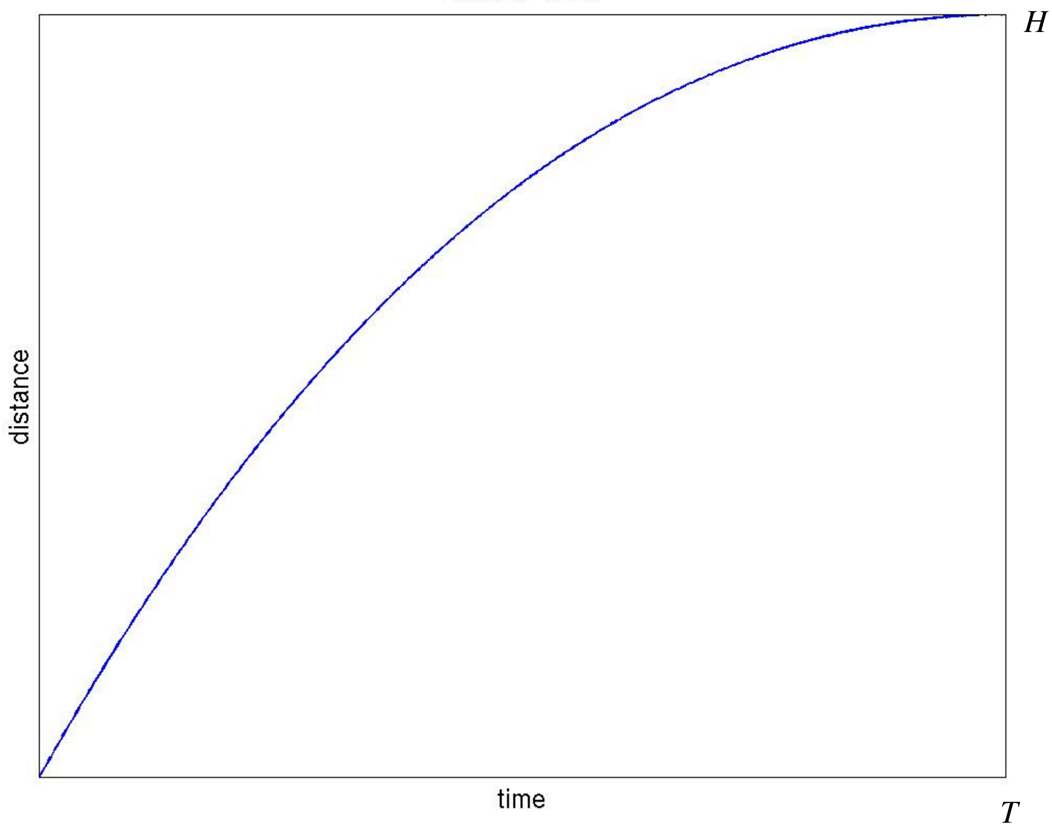
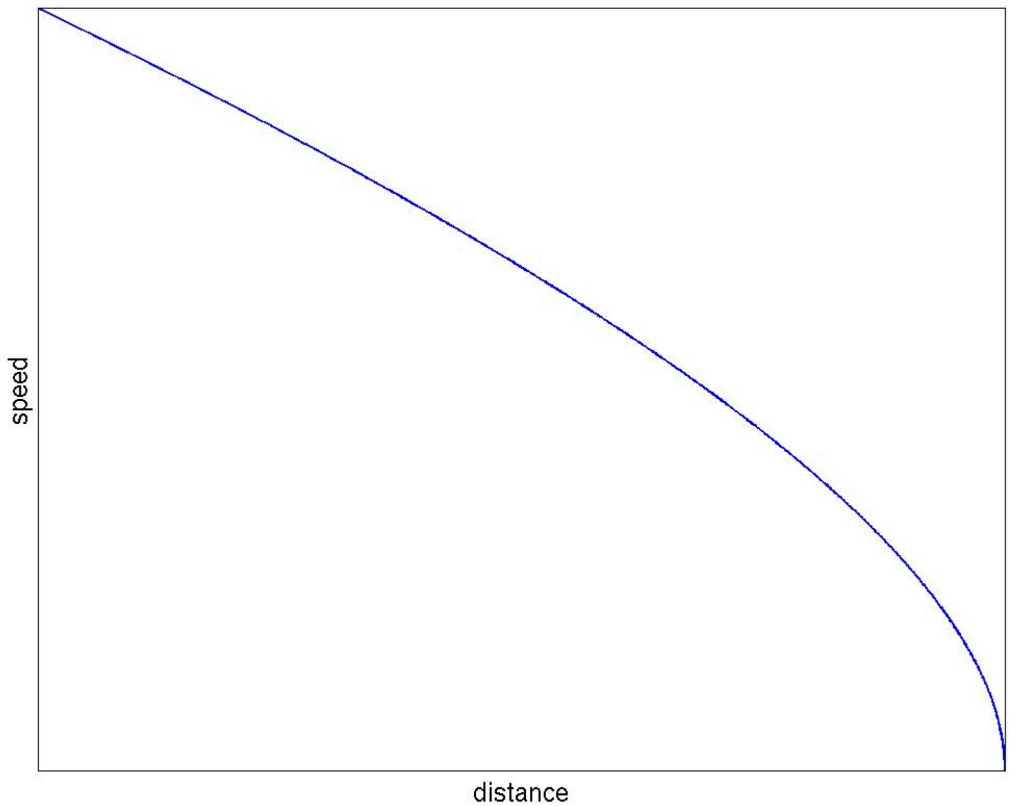


FIGURE VI.11



It might be remarked that you cannot, on this occasion, obtain the space integral by algebraic elimination of t between the two time integrals. You should verify that, for small t, $\upsilon \approx \upsilon_0 - gt$ and $y \approx \upsilon_0 t$; and for small y, $\upsilon^2 \approx \upsilon_0^2 - 2gy$.

The maximum height H is reached when $\upsilon = 0$ in equation 6.3.28:

$$H = \frac{1}{\gamma}\left[\upsilon_0 - \hat{\upsilon}\ln\left(1+\frac{\upsilon_0}{\hat{\upsilon}}\right)\right]. \qquad 6.3.29$$

For a puck sliding along ice and subject to air resistance and a frictional force, the equation of motion is

$$\ddot{x} = -\mu g - \gamma\dot{x}. \qquad 6.3.30$$

This is very similar to equation 6.3.24, with the substitution of μg for g, and all the subsequent equations and conclusions follow, including the maximum distance travelled and the time taken to get there – except that the equations seem to predict that, when the speed becomes zero, the acceleration is $-\mu g$, and the puck will subsequently start to move backwards (just as the ball thrown vertically upwards will start to fall). The reader is invited to cogitate upon this and determine just where there is a difference between the motion of the one and the motion of the other.

More problems on motion in which the resistance is proportional to the speed

5. A particle is projected upwards. The average speed between launch and the highest point is one third of the speed of projection. Calculate the ratio of the terminal speed to the speed of projection.

6. A particle is projected upwards. Show that the average speed between launch and the highest point cannot be more than half the speed of projection.

7. A particle is projected upwards in a medium of damping constant 1.2 s^{-1}. It takes twice as long to come down as to go up. What was the initial speed? (g = 9.8 m s^{-2}.)

8. A particle falls from rest. The damping constant is γ and the gravitational acceleration is g.
   Expand y as a power series in t up to t^3.
   Expand y as a power series in υ up to υ^3.

9. Two identical particles are moving in a medium in which their terminal speed is $\hat{\upsilon}$. One is launched vertically upwards from the ground with initial speed $\hat{\upsilon}$. At the same instant, the other is launched vertically downwards, also with initial speed $\hat{\upsilon}$, from a point at a vertical height h above the point of projection of the first particle. Show that, provided that $2\hat{\upsilon}^2 > gh$, the two particles meet after a time

$$-\frac{\hat{\upsilon}}{g}\ln\left(1-\frac{gh}{2\hat{\upsilon}^2}\right).$$

10. A particle is launched vertically upwards with a speed U times the terminal speed. Show that on return it hits the ground with a speed V times the terminal speed, where

$$U+V = \ln\left(\frac{1+U}{1-V}\right).$$

Draw a graph of V (vertical axis) versus U (horizontal axis) for U going from 0 to 2.

Answers. **5**. $\hat{\upsilon}/\upsilon_0$ = 0.1320. **7**. 29.96 m s^{-1} **8**. $y = \frac{1}{2}gt^2 - \frac{1}{6}g\gamma t^3$; $y = \frac{\upsilon^2}{2g} + \frac{\gamma\upsilon^3}{3g^2}$.

11. A particle is launched vertically upwards at an initial speed $\upsilon_0 = 60$ m s^{-1}. The air resistance is proportional to the speed. The latitude is such that $g = 9.8$ m s^{-2}. It takes twice as long to fall from its maximum height as it takes to reach its maximum height. How long does the total journey take, what is the maximum height, and what is its speed when it returns to the ground?

I found this problem a little harder than I expected. Here is my solution.

Notation: H = maximum height
T_1 = time to reach it
$T_2 = 2T_1$ = time to fall back to the ground.
g = acceleration due to gravity
γ = damping constant
υ_0 = initial speed

According to equation 6.3.29

$$H = \frac{1}{\gamma}\left[\upsilon_0 - \frac{g}{\gamma}\ln\left(1 + \frac{\gamma\upsilon_0}{g}\right)\right]. \tag{1}$$

According to equation 6.3.26

$$T_1 = \frac{1}{\gamma}\ln\left(1 + \frac{\gamma\upsilon_0}{g}\right). \tag{2}$$

The statement of the problem tells us that

$$T_2 = \frac{2}{\gamma}\ln\left(1 + \frac{\gamma\upsilon_0}{g}\right). \tag{3}$$

According to equation 6.3.14

$$H = \frac{gT_2}{\gamma} - \frac{g}{\gamma^2}\left(1 - e^{-\gamma T_2}\right). \tag{4}$$

Substitute equation (3) into equation (4), and equate the resulting expression for H to the expression given by equation (1). This is facilitated a little by writing $x = 1 + \frac{\gamma\upsilon_0}{g}$.

After a little algebra, I obtain

$$3\ln x - x + \frac{1}{x^2} = 0. \tag{5}$$

This has an obvious solution $x = 1$, which implies that $\upsilon_0 = 0$, which is a little disconcerting, though it certainly satisfies the problem in that the return time is twice the outgoing time. (Both are zero!) However, there is another solution, namely $x = 4.6682236$, which immediately tells us that the damping constant is $\gamma = 0.59914319\ \mathrm{s}^{-1}$. Then equation (2), which is $T_1 = \frac{\ln x}{\gamma}$, gives us that $T_1 = 2.5716367$ s, so the total journey time is 7.7149101 s. The maximum height reached is given by equation (1) or (4). Do both as a check on the algebra and arithmetic. From either of these I get $H = 58.079538$ m. Finally, we can get the speed of landing, υ_2, from equation 6.3.12:

$$\upsilon_2 = \frac{g}{\gamma}\left(1 - e^{-\gamma T_2}\right) \tag{6}$$

or from equation 6.3.16:

$$H = -\frac{g}{\gamma^2}\ln\left(1 - \frac{\gamma\upsilon_2}{g}\right) - \frac{\upsilon_2}{\gamma}. \tag{7}$$

It is easy to calculate υ_2 from equation (6). It is a bit bothersome to do it from equation (7), but our check will be satisfied if we calculate υ_2 from equation (6) and then substitute it into equation (7) to verify that we get the correct value for H. I make it that $\upsilon_2 = 15.606119\ \mathrm{m\ s}^{-1}$.

6.4 *Motion in which the Resistance is Proportional to the Square of the Speed.*

There are not really any new principles; it is just a matter of practice with slightly more difficult integrals. We assume that the resistive force per unit mass is $k\dot{x}^2$. Here, although k plays a somewhat similar role to the γ of section 3, it is not exactly the same thing as γ, and indeed it is not dimensionally the same as γ. What are the dimensions, and the SI units, of k?

6.4a. *Resistive force only*

We'll imagine a puck sliding along a frictionless surface against turbulent air resistance. The equation of motion is:

$$\ddot{x} = -kv^2. \qquad 6.4.1$$

By this time we assume that the student knows how to obtain the first and second time integrals and the space integral. The actual integrations may be slightly more difficult, but we leave it to the reader to obtain the results

$$v = \frac{v_0}{1+kv_0t}, \qquad 6.4.2$$

$$x = \frac{\ln(1+kv_0t)}{k}, \qquad 6.4.3$$

$$v = v_0e^{-kx}. \qquad 6.4.4$$

These are illustrated in figures VI.12,13,14. Note that, provided that equation 6.4.1 accurately describes the entire motion (which may not be the case in a practical situation), there is no finite limit to x, nor does the speed drop to zero in any finite time.

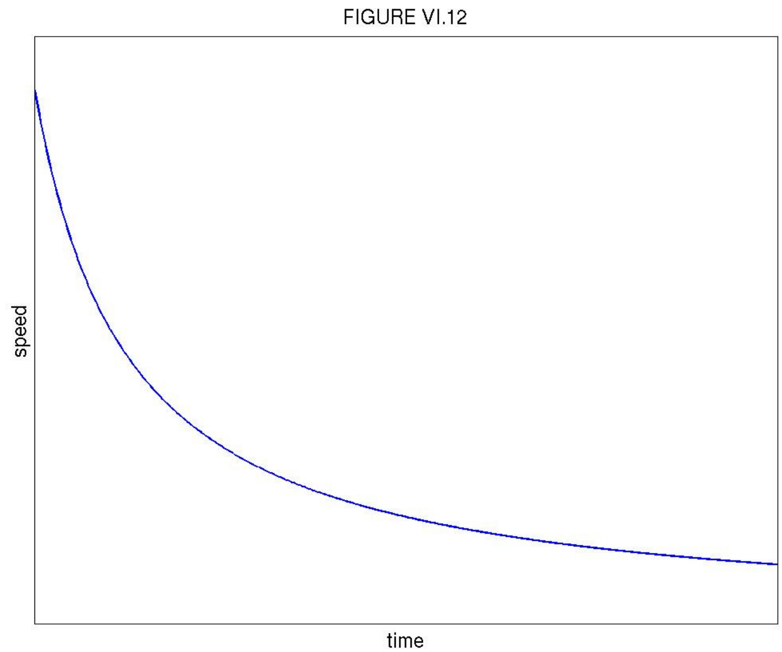


FIGURE VI.12
speed
time

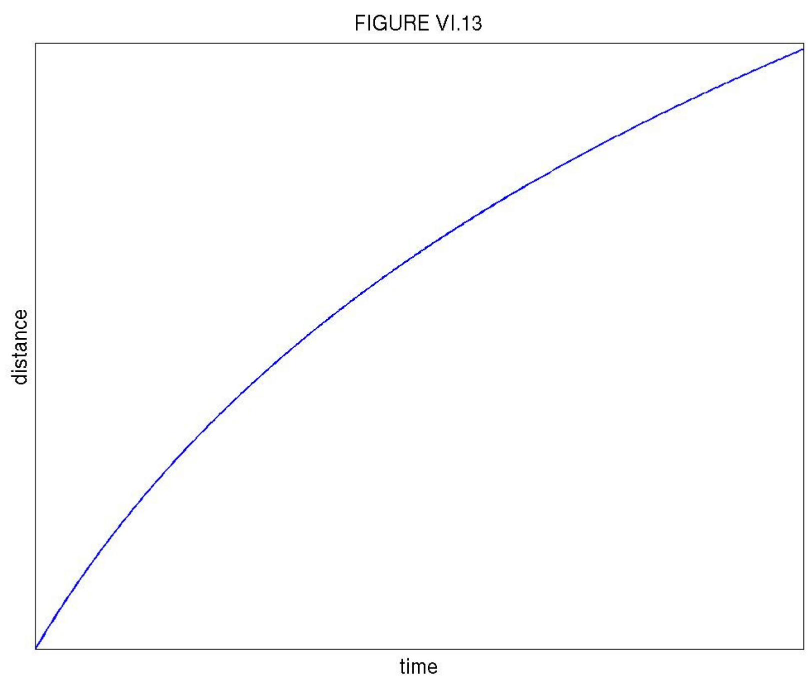
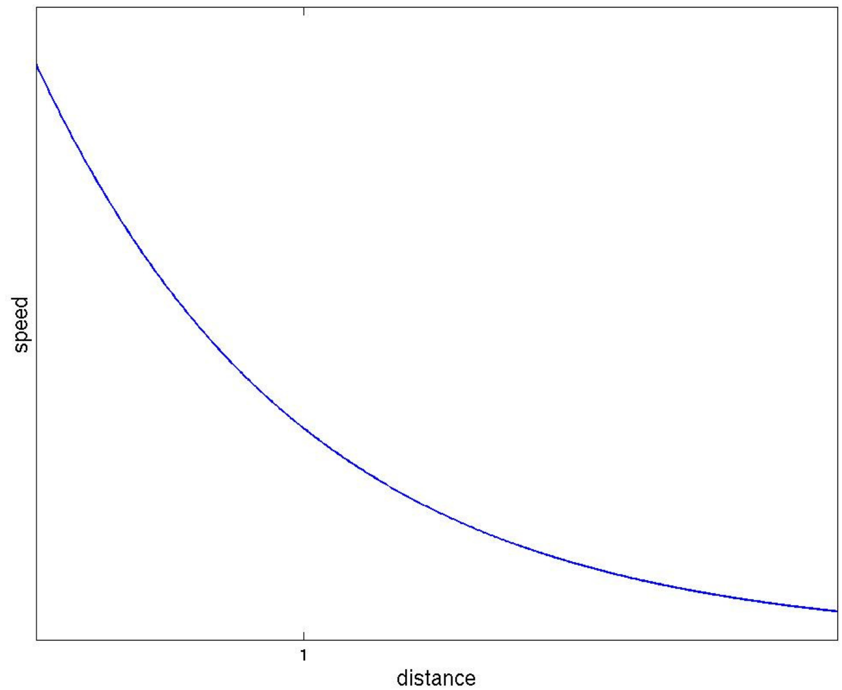


FIGURE VI.13
distance
time

FIGURE VI.14



6.4b. *Body falling under gravity in a resisting medium, resistive force proportional to the square of the speed*

The equation of motion is

$$\ddot{y} = g - kv^2. \tag{6.4.4}$$

The limiting, or terminal, speed, which is reached when the acceleration is zero, is given by

$$\hat{v} = \sqrt{g/k}, \tag{6.4.5}$$

so that the equation of motion can be written

$$\ddot{y} = k\left(\hat{v}^2 - v^2\right). \tag{6.4.6}$$

The first and second time integrals and the space integral can be found in the usual way, though with perhaps a little more effort:

$$v = \hat{v}\tanh\left(k\hat{v}t\right), \tag{6.4.7}$$

$$y = \frac{\ln[\cosh(k\hat{\upsilon}t)]}{k}, \qquad 6.4.8$$

$$\upsilon^2 = \hat{\upsilon}^2\left(1 - e^{-2ky}\right). \qquad 6.4.9$$

These are illustrated in figures VI.15, 16,17. The inverse relations corresponding to equations 6.4.7 and 6.4.8 can be found from the usual formulas for the inverse hyperbolic functions $\tanh^{-1} x = \ln\sqrt{\frac{1+x}{1-x}}$ and $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$.

FIGURE VI.15

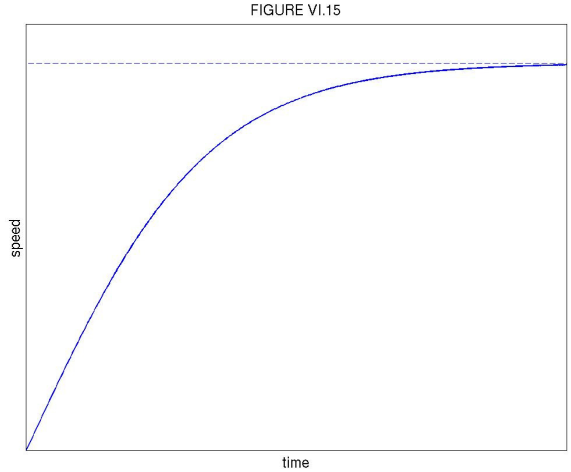


FIGURE VI.16

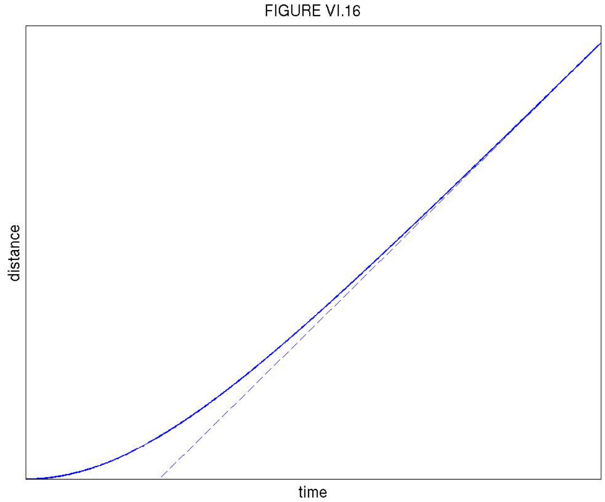
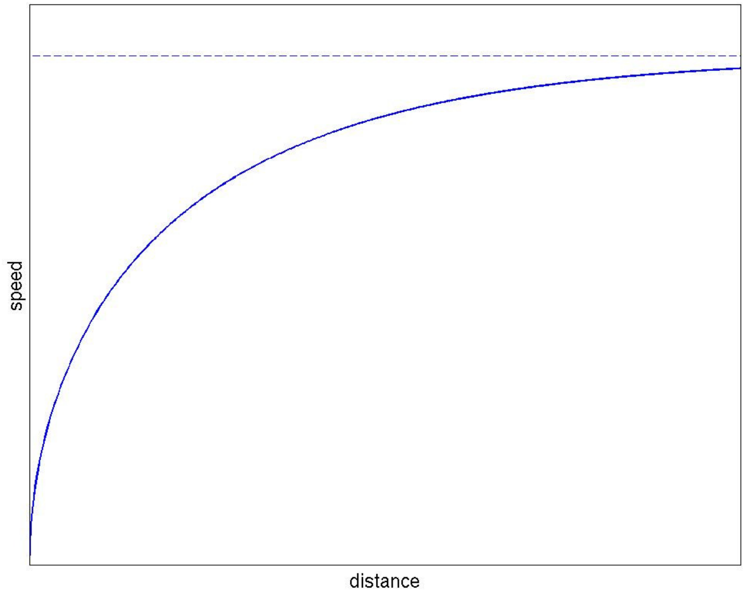


FIGURE VI.17



6.4c. *Body thrown vertically upwards, initial speed* υ_0 , *resistance proportional to the square of the speed.*

The equation of motion is

$$\ddot{y} = -\left(g + k\upsilon^2\right) = -k\left(\hat{\upsilon}^2 + \upsilon^2\right), \qquad 6.4.10$$

where $\hat{\upsilon} = \sqrt{g/k}$. That is to say

$$\int \frac{d\upsilon}{\hat{\upsilon}^2 + \upsilon^2} = -\int k dt. \qquad 6.4.11$$

The *first time integral* is

$$\frac{1}{\hat{\upsilon}} \tan^{-1} \frac{\upsilon}{\hat{\upsilon}} = \text{constant} - kt. \qquad 6.4.12$$

Since the constant of integration has dimensions of the reciprocal of a speed, I shall use the symbol $\frac{1}{V}$ for the constant, so that the first time integral becomes

$$\upsilon = \hat{\upsilon} \tan\left(\frac{\hat{\upsilon}}{V} - k\hat{\upsilon}t\right). \qquad 6.4.13$$

The constant V can be evaluated from the initial condition that $\upsilon = \upsilon_0$ at $t = 0$, so that

$$\frac{\hat{\upsilon}}{V} = \tan^{-1}\frac{\upsilon_0}{\hat{\upsilon}}. \qquad 6.4.14$$

The time T to maximum height is found by putting $\upsilon = 0,\ t = T$ in equation 6.4.13 to obtain

$$T = \frac{1}{Vk} \quad . \qquad 6.4.15$$

The *second time integral* is found by writing $\frac{dy}{dt}$ for υ in equation 6.4.13 and integrating, to obtain:

$$y \ = \ \frac{1}{k}\ln\left[\frac{\cos\left(\frac{\hat{\upsilon}}{V} - k\hat{\upsilon}t\right)}{\cos\frac{\hat{\upsilon}}{V}}\right]. \qquad 6.4.16$$

The maximum height reached is found by substituting T for t and H for y in equation 6.4.16 to obtain

$$H \ = \ \frac{1}{k}\ln\sec\frac{\hat{\upsilon}}{V}. \qquad 6.4.17$$

The space integral is found by writing $\upsilon\frac{d\upsilon}{dy}$ for $\ddot{y}$ in equation 6.4.10 and integrating, to obtain

$$\upsilon^2 \ = \left(\upsilon_0^2 + \hat{\upsilon}^2\right)e^{-2ky} - \hat{\upsilon}^2. \qquad 6.4.18$$

The maximum height H is reached when the speed is zero, whence

$$H \ = \frac{1}{2k}\ln\left(1 + \frac{\upsilon_0^2}{\hat{\upsilon}^2}\right). \qquad 6.4.19$$

It is easy to show (by making use of equation 6.4.14, together with a little help from Pythagoras) that equations 6.4.17 and 6.4.19 are equivalent.

Let us write these equations in a form such that

acceleration is expressed in units of g,
speed is expressed in units of $\sqrt{g/k}$,
distance is expressed in units of $1/k$, and
time is expressed in units of $1/\sqrt{gk}$

In that case, the equations become as follows.

The equation of motion, 6.4.10, becomes

$$\ddot{y} = 1 + \upsilon^2 . \qquad 6.4.19$$

The first time integral, 6.4.13, which gives speed as a function of time, combined with 6.4.15, which gives the integration constant, becomes

$$\upsilon = \frac{\upsilon_0 - \tan t}{1 + \upsilon_0 \tan t}. \qquad 6.4.20a$$

The converse of this equation is

$$t = \tan^{-1}\upsilon_0 - \tan^{-1}\upsilon \qquad 6.4.20b$$

The time T to maximum height, which occurs when $\upsilon = 0$, becomes

$$T = \tan^{-1}\upsilon_0 . \qquad 6.4.21$$

The second time integral, 6.4.16, which gives distance as a function of time, combined with 6.4.15, which gives the integration constant, becomes

$$y = \ln(\cos t + \upsilon_0 \sin t) . \qquad 6.4.22a$$

The converse of this equation is

$$t = \sin^{-1}\left(\frac{e^y}{\sqrt{1+\upsilon_0^2}}\right) - \cot^{-1}\upsilon_0. \qquad 6.4.22b$$

The maximum height H is calculated by substituting H for y, and T for t in equation 6.4 22.

$$H = \ln\sqrt{1+\upsilon_0^2} . \qquad 6.4.23$$

The space integral, 6.4.18, which gives speed as a function of distance, becomes

$$\upsilon^2 = (\upsilon_0^2 + 1)e^{-2y} - 1. \qquad 6.4.22$$

The maximum height *H* occurs where $\upsilon = 0$, which results again in equation 6.4.23, in which *H* was calculated through a different route.

$$H = \ln\sqrt{1 + \upsilon_0^2}. \qquad 6.4.23$$

Numerical example. Let us suppose that we project a body vertically upwards, the resistance being proportional to the square of the speed, with an initial speed of $\upsilon_0 = 2$, in units of $\sqrt{g/k}$. It will reach a maximum height (in units of 1/*k*) of $H = \ln\sqrt{5} = 0.80472$ in time $T = \tan^{-1} 2 = 1.10715$ (in units of $1/\sqrt{gk}$). Thereafter it falls, taking time $\cosh^{-1} H = \ln(\sqrt{5} + 2) = 1.44364$ to reach the ground. The total time of flight is 2.55079. Figure VI.18 shows the speed (magnitude of the velocity) as a function of time (first time integral) as the body at first rises and then falls. The terminal speed (not reached at ground level - or indeed in any finite time) is, of course, 1. The speed when it hits the ground is $\sqrt{0.8} = 89443$. The value of *V*, which was a constant of integration and, as far as I can see, has no particular physical significance, is $1/\tan^{-1} 2 = 0.90322$, and it occurs at time $\tan^{-1}\left(\frac{2 - V}{1 + 2V}\right) = 0.37256$, when the height is $\ln\sqrt{\frac{5}{1 + V^2}} = 0.50645$.

FIGURE VI.18

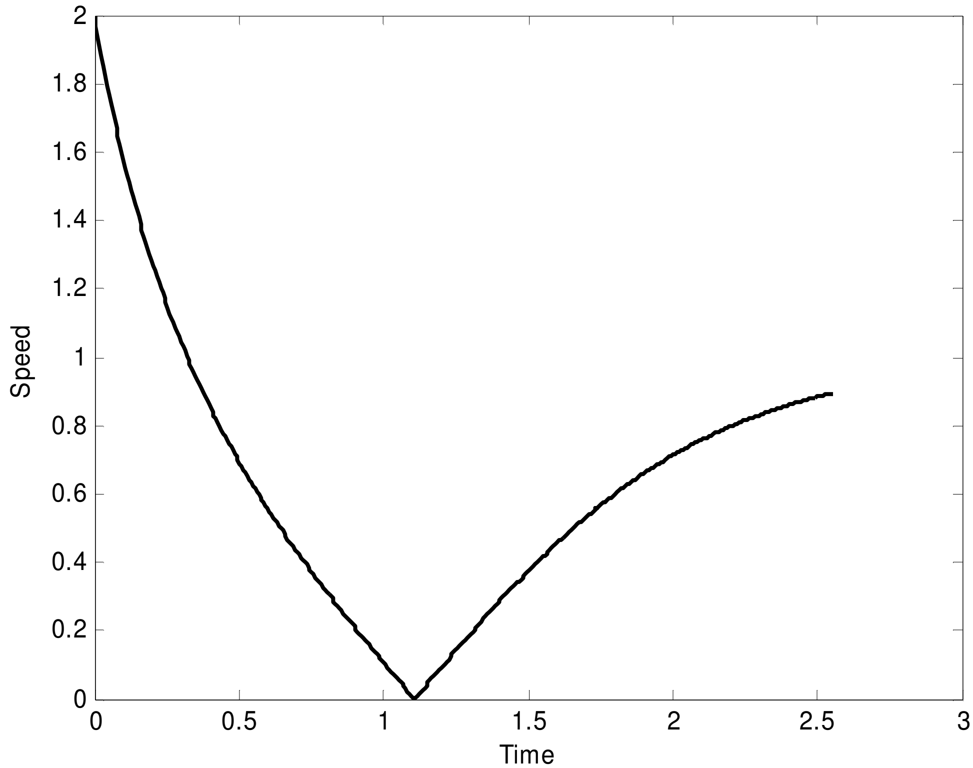


Figure VI.19 shows the height as a function of time (second time integral).

FIGURE VI.19

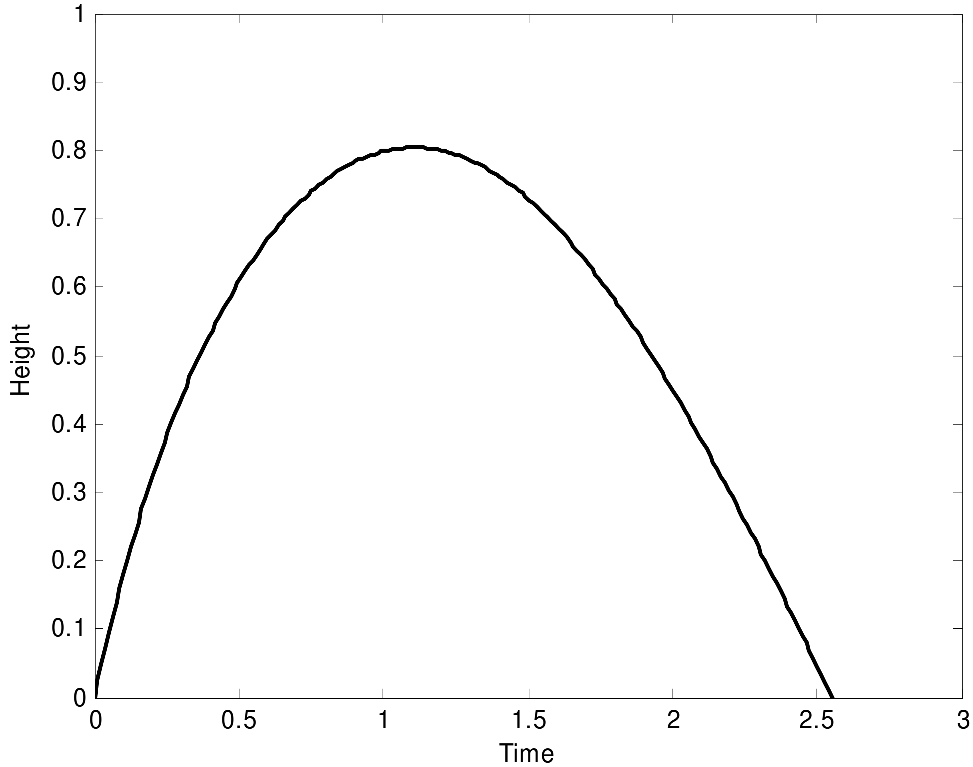
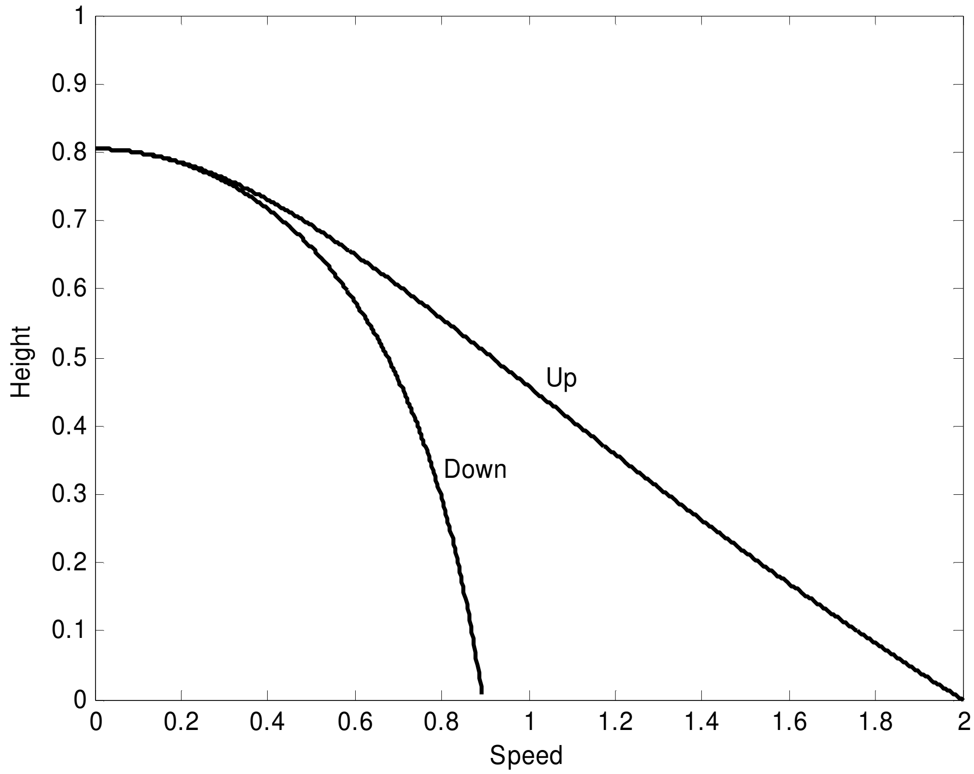


Figure VI.20 shows the speed as a function of height (space integral).

FIGURE VI.20



Problem

A particle is projected vertically upward with initial speed equal to tan α times the terminal speed, the resistance being proportional to the square of the speed. Show that on return the particle hits the ground with speed sin α times the terminal speed. For example, in our numerical example, the initial speed was twice the terminal speed, corresponding to $\alpha = \tan^{-1} 2 = 1.10715$. If the result asked for in this problem is correct, the speed on reaching the ground should be $\sin\alpha = 0.89443$, as indeed it is seen to be in figure VI.20.