CHAPTER 3
SYSTEMS OF PARTICLES

3.1 *Introduction*

By systems of particles I mean such things as a swarm of bees, a star cluster, a cloud of gas, an atom, a brick. A brick is indeed composed of a system of particles – atoms – which are constrained so that there is very little motion (apart from small amplitude vibrations) of the particles relative to each other. In a system of particles there may be very little or no interaction between the particles (as in a loose association of stars separated from each other by large distances) or there may be (as in the brick) strong forces between the particles. Most (perhaps all) of the results to be derived in this chapter for a system of particles apply equally to an apparently solid body such as a brick. Even if scientists are wrong and a brick is not composed of atoms but is a genuine continuous solid, we can in our imagination suppose the brick to be made up of an infinite number of infinitesimal mass and volume elements, and the same results will apply.

What sort of properties shall we be discussing? Perhaps the simplest one is this: *The total linear momentum of a system of particles is equal to the total mass times the velocity of the center of mass.* This is true, and it may be "obvious" – but it still requires proof. It may be equally "obvious" to some that "the total kinetic energy of a system of particles is equal to $\frac{1}{2}M\overline{v}^2$, where M is the total mass and $\overline{v}$ is the velocity of the center of mass" – but this one, however "obvious", is <u>not</u> true!

Before we get round to properties of systems of particles, I want to clarify what I mean by the *moment* of a vector such as a force or momentum. You are already familiar, from Chapters 1 and 2, with the moments of *mass*, which is a scalar quantity.

3.2 *Moment of a Force*

First, let's look at a familiar two-dimensional situation. In figure III.1 I draw a force **F** and a point O. The moment of the force with respect to O can be defined as
<u>Force times perpendicular distance from O to the line of action of **F**.</u>

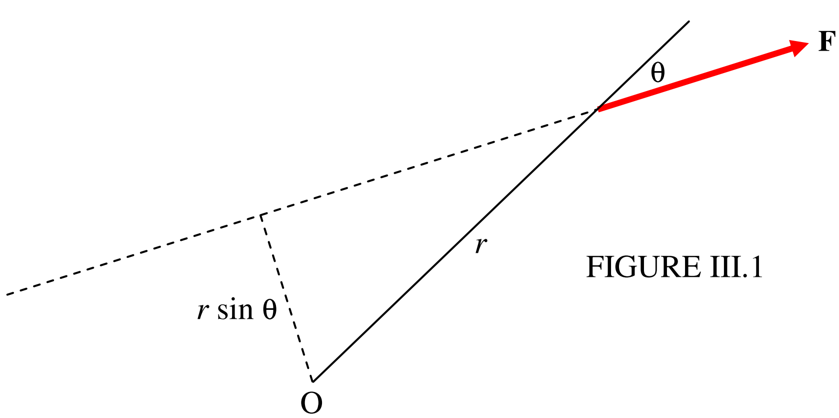

FIGURE III.1

Alternatively, (figure III.2) the moment can be defined equally well by
Transverse component of force times distance from O to the point of application of the force.

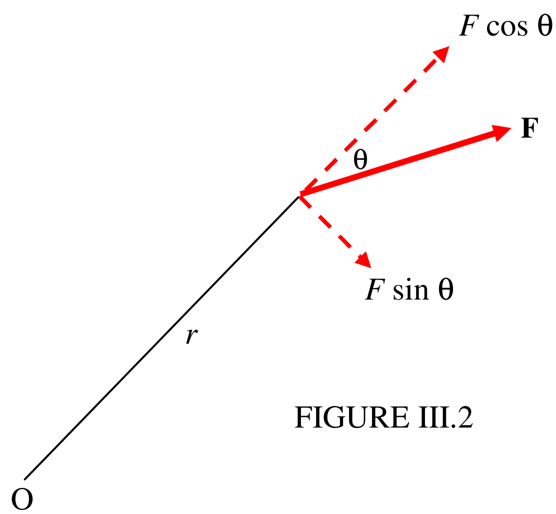

FIGURE III.2

Either way, the magnitude of the moment of the force, also known as the *torque*, is $rF\sin\theta$. We can regard it as a vector, $\boldsymbol{\tau}$, perpendicular to the plane of the paper:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}. \qquad 3.2.1$$

Now let me ask a question. Is it correct to say the moment of a force with respect to (or "about") a *point* or with respect to (or "about") an *axis*?

In the above two-dimensional example, it does not matter, but now let me move on to three dimensions, and I shall try to clarify.

In figure III.3, I draw a set of rectangular axes, and a force **F**, whose position vector with respect to the origin is **r**.

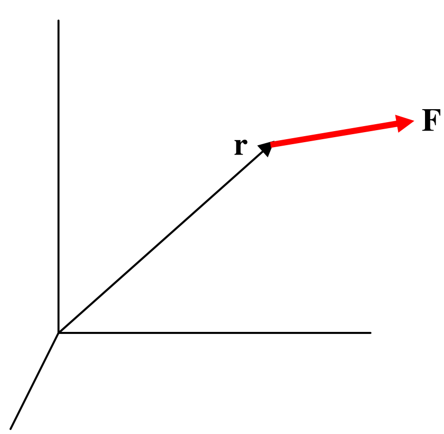

FIGURE III.3

The moment, or *torque*, of **F** *with respect to the origin* is the vector

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}. \qquad 3.2.2$$

The *x*-, *y*- and *z-components* of $\boldsymbol{\tau}$ are the moments of **F** *with respect to the x-, y- and z-axes.* You can easily find the components of $\boldsymbol{\tau}$ by expanding the cross product 3.2.2:

$$\boldsymbol{\tau} = \hat{\mathbf{x}}(yF_z - zF_y) + \hat{\mathbf{y}}(zF_x - xF_z) + \hat{\mathbf{z}}(xF_y - yF_x), \qquad 3.2.3$$

where $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are the unit vectors along the *x*, *y*, z axes. In figure III.4, we are looking down the *x*-axis, and I have drawn the components F_y and F_z, and you can see that, indeed, $\tau_x = yF_z - zF_y$.

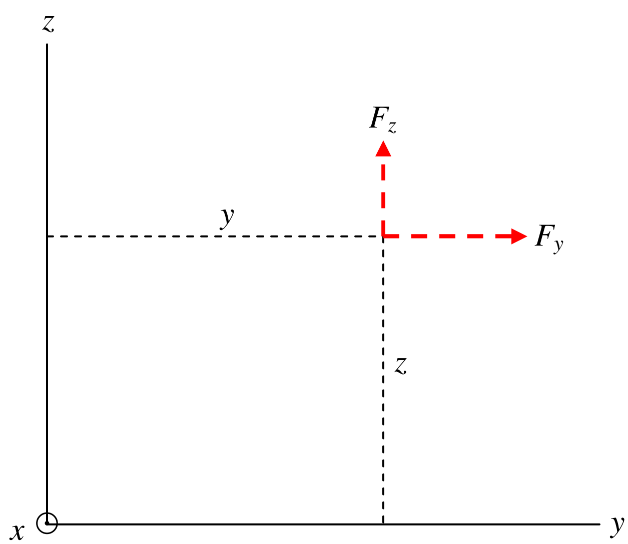

FIGURE III.4

The dimensions of moment of a force, or torque, are ML^2T^{-2}, and the SI units are N m. (It is best to leave the units as N m rather than to express torque in joules.)

3.3 *Moment of Momentum*

In a similar way, if a particle at position **r** has linear momentum $\mathbf{p} = m\mathbf{v}$, its *moment of momentum with respect to the origin is the vector* **l** *defined by*

$$\mathbf{l} = \mathbf{r} \times \mathbf{p}, \qquad 3.3.1$$

and its *components* are the moments of momentum *with respect to the axes.* Moment of momentum plays a role in rotational motion analogous to the role played by linear momentum in linear motion, and is also called *angular momentum.* The dimensions of angular momentum are ML^2T^{-1}. Several choices for expressing angular momentum in SI units are possible; the usual choice is J s (joule seconds).

3.4 *Notation*

In this section I am going to suppose that we n particles scattered through three-dimensional space. We shall be deriving some general properties and theorems – and, to the extent that a solid body can be considered to be made up of a system of particles, these properties and theorems will apply equally to a solid body.

In the figure III.5, I have drawn just two of the particles, (the rest of them are left to your imagination) and the centre of mass C of the system.

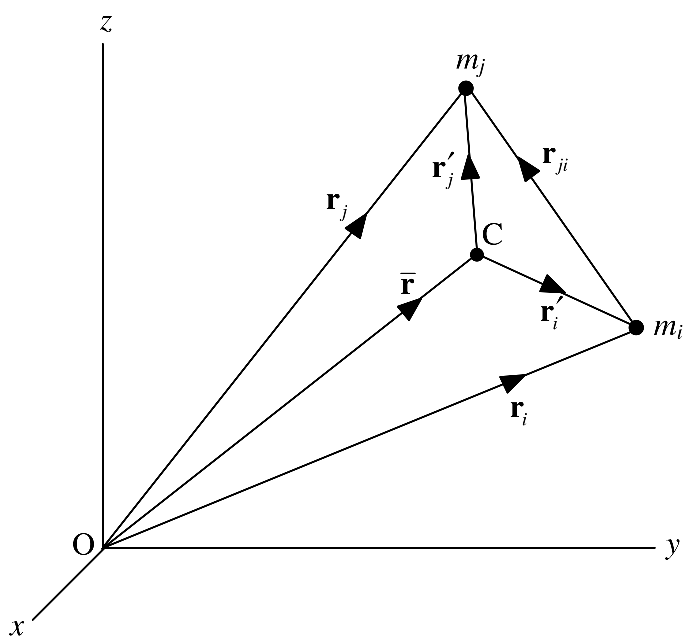

FIGURE III.5

A given particle may have an *external force* $\mathbf{F}_i$ acting upon it. (It may, of course, have *several* external forces acting on it, but I mean by $\mathbf{F}_i$ the vector sum of all the external forces acting on the ith particle.) It may also interact with the other particles in the system, and consequently it may have *internal forces* $\mathbf{F}_{ij}$ acting upon it, where j goes from 1 to n except for i. I define the vector sum $\mathbf{F} = \sum \mathbf{F}_i$ as the total external force acting upon the *system*.

I am going to establish the following notation for the purposes of this chapter.

Mass of the ith particle $= m_i$

Total mass of the system $M = \sum m_i$

Position vector of the ith particle referred to a fixed point O: $\mathbf{r}_i = x_i\hat{\mathbf{x}} + y_i\hat{\mathbf{y}} + z_i\hat{\mathbf{z}}$

Velocity of the ith particle referred to a fixed point O: $\dot{\mathbf{r}}_i$ or $\mathbf{v}_i$ (Speed = υ_i)

Linear momentum of the ith particle referred to a fixed point O: $\mathbf{p}_i = m_i\mathbf{v}_i$

Linear momentum of the *system*: $\mathbf{P} = \sum \mathbf{p}_i = \sum m_i\mathbf{v}_i$

External force on the ith particle: $\mathbf{F}_i$

Total external force on the system: $\mathbf{F} = \sum \mathbf{F}_i$

Angular momentum (moment of momentum) of the ith particle referred to a fixed point O:

$$\mathbf{l}_i = \mathbf{r}_i \times \mathbf{p}_i$$

Angular momentum of the system: $\mathbf{L} = \sum \mathbf{l}_i = \sum \mathbf{r}_i \times \mathbf{p}_i$

Torque on the ith particle referred to a fixed point O: $\boldsymbol{\tau}_i = \mathbf{r}_i \times \mathbf{F}_i$

Total external torque on the system with respect to the origin:

$$\boldsymbol{\tau} = \sum \boldsymbol{\tau}_i = \sum \mathbf{r}_i \times \mathbf{F}_i\,.$$

Kinetic energy of the system: (We are dealing with a system of *particles* – so we are dealing only with *translational* kinetic energy – no rotation or vibration):

$$T = \sum \tfrac{1}{2} m_i \upsilon_i^2$$

Position vector of the *centre of mass* referred to a fixed point O: $\bar{\mathbf{r}}_i = \bar{x}\hat{\mathbf{x}} + \bar{y}\hat{\mathbf{y}} + \bar{z}\hat{\mathbf{z}}$

The centre of mass is defined by $M\bar{\mathbf{r}} = \sum m_i\mathbf{r}_i$

Velocity of the centre of mass referred to a fixed point O: $\dot{\bar{\mathbf{r}}}$ or $\bar{\mathbf{v}}$ (Speed = $\bar{\upsilon}$)

For position vectors, unprimed single-subscript symbols will refer to O. Primed single-subscript symbols will refer to C. This will be clear, I hope, from figure III.5.

Position vector of the *i*th particle *referred to the centre of mass* C: $\mathbf{r'}_i = \mathbf{r}_i - \overline{\mathbf{r}}_i$

Position vector of particle *j* with respect to particle *i*: $\mathbf{r}_{ji} = \mathbf{r}_j - \mathbf{r}_i$

(Internal) force exerted on particle *i* by particle *j*: $\mathbf{F}_{ij}$

(Internal) force exerted on particle *j* by particle *i*: $\mathbf{F}_{ji}$

If the force between two particles is *repulsive* (e.g. between electrically-charged particles of the same sign), then $\mathbf{F}_{ji}$ and $\mathbf{r}_{ji}$ are in the same direction. But if the force is an *attractive* force, $\mathbf{F}_{ji}$ and $\mathbf{r}_{ji}$ are in opposite directions.

According to Newton's Third Law of Motion (Lex III), $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$

Total angular momentum of system referred to the centre of mass C: $\mathbf{L}_\mathrm{C}$

Total external torque on system referred to the centre of mass C: $\boldsymbol{\tau}_\mathrm{C}$

For the velocity of the centre of mass I may use either $\dot{\overline{\mathbf{r}}}$ or $\overline{\mathbf{v}}$.

O is an arbitrary origin of coordinates. C is the centre of mass.

Note that $$\mathbf{r}_i = \overline{\mathbf{r}} + \mathbf{r}'_i \qquad 3.4.1$$

and therefore $$\dot{\mathbf{r}}_i = \dot{\overline{\mathbf{r}}} + \dot{\mathbf{r}}'_i \;; \qquad 3.4.2$$

that is to say $$\mathbf{v}_i = \overline{\mathbf{v}} + \mathbf{v}'_i . \qquad 3.4.3$$

Note also that $$\sum m_i \mathbf{r}'_i = 0. \qquad 3.4.4$$

Note further that

$$\sum m_i \mathbf{v}'_i = \sum m_i (\mathbf{v}_i - \overline{\mathbf{v}}) = \sum m_i \mathbf{v}_i - \overline{\mathbf{v}} \sum m_i = M\overline{\mathbf{v}} - \overline{\mathbf{v}}M = 0. \qquad 3.4.5$$

That is, *the total linear momentum with respect to the centre of mass is zero.*

Having established our notation, we now move on to some theorems concerning systems of particles. It may be more useful for you to conjure up a physical picture in your mind what the following theorems mean than to memorize the details of the derivations.

3.5 *Linear Momentum*

Theorem: *The total momentum of a system of particles equals the total mass times the velocity of the centre of mass.*

Thus: $$\mathbf{P} = \sum m_i \mathbf{v}_i = \sum m_i(\overline{\mathbf{v}} + \mathbf{v}'_i) = M\overline{\mathbf{v}} + 0. \qquad 3.5.1$$

3.6 *Force and Rate of Change of Momentum*

Theorem: *The rate of change of the total momentum of a system of particles is equal to the sum of the external forces on the system.*

Thus, consider a single particle. By Newton's second law of motion, the rate of change of momentum of the particle is equal to the sum of the forces acting upon it:

$$\dot{\mathbf{p}}_i = \mathbf{F}_i + \sum_j \mathbf{F}_{ij}. \qquad (j \neq i) \qquad 3.6.1$$

Now sum over all the particles:

$$\dot{\mathbf{P}} = \sum_i \mathbf{F}_i + \sum_i \sum_j \mathbf{F}_{ij} \qquad (j \neq i)$$

$$= \mathbf{F} + \tfrac{1}{2}\sum_i \sum_j \mathbf{F}_{ji} + \tfrac{1}{2}\sum_j \sum_i \mathbf{F}_{ij}$$

$$= \mathbf{F} + \tfrac{1}{2}\sum_i \sum_j (\mathbf{F}_{ji} + \mathbf{F}_{ij}). \qquad 3.6.2$$

But, by Newton's third law of motion, $\mathbf{F}_{ji} + \mathbf{F}_{ij} = 0$, so the theorem is proved.

Corollary: If the sum of the external forces on a system is zero, the linear momentum is constant. (Law of Conservation of Linear Momentum.)

3.7 *Angular Momentum*

Notation: $\mathbf{L}_C$ = angular momentum of system with respect to centre of mass C.
$\mathbf{L}$ = angular momentum of system relative to some other origin O.
$\overline{\mathbf{r}}$ = position vector of C with respect to O.

$\mathbf{P}$ = linear momentum of system with respect to O.
(The linear momentum with respect to C is, of course, zero.)

Theorem: $$\mathbf{L} = \mathbf{L}_\mathrm{C} + \bar{\mathbf{r}}\times\mathbf{P}. \qquad 3.7.1$$

Thus: $$\mathbf{L} = \sum \mathbf{r}_i \times \mathbf{p}_i = \sum m_i(\mathbf{r}_i \times \mathbf{v}_i) = \sum m_i(\bar{\mathbf{r}} + \mathbf{r'}_i)\times(\bar{\mathbf{v}} + \mathbf{v'}_i)$$

$$= (\bar{\mathbf{r}}\times\bar{\mathbf{v}})\sum m_i + \bar{\mathbf{r}}\times\sum m_i\mathbf{v'}_i + \left(\sum m_i\mathbf{r'}_i\right)\times\bar{\mathbf{v}} + \sum \mathbf{r'}_i\times\mathbf{p'}_i$$

$$= M(\bar{\mathbf{r}}\times\bar{\mathbf{v}}) + \bar{\mathbf{r}}\times 0 + 0\times\bar{\mathbf{v}} + \mathbf{L}_\mathrm{C}\,.$$

$$\therefore \qquad \mathbf{L} = \mathbf{L}_\mathrm{C} + \bar{\mathbf{r}}\times\mathbf{P}.$$

Example. A hoop of radius a rolling along the ground (figure III.6):

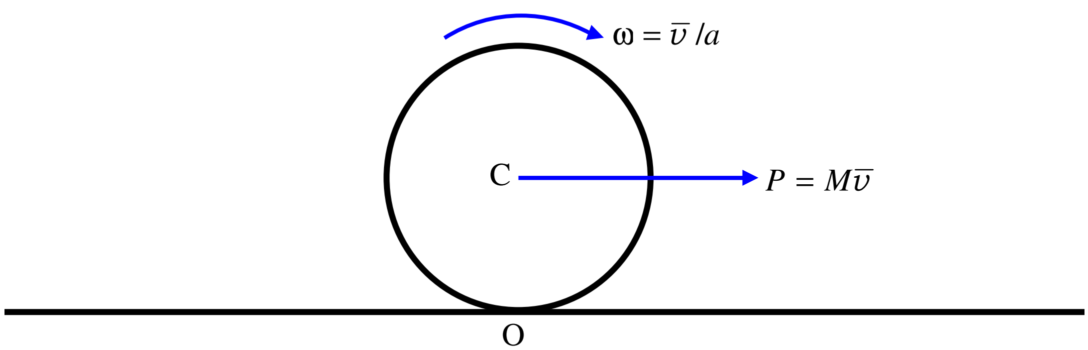

FIGURE III.6

The angular momentum with respect to C is $L_\mathrm{C} = I_\mathrm{C}\omega$, where I_C is the rotational inertia about C. The angular momentum about O is therefore

$$L = I_\mathrm{C}\omega + M\bar{v}a = I_\mathrm{C}\omega + Ma^2\omega = (I_\mathrm{C} + Ma^2) = I\omega,$$

where $I = I_\mathrm{C} + Ma^2$ is the rotational inertia about O.

3.8 *Torque*

Notation: $\boldsymbol{\tau}_\mathrm{C}$ = vector sum of all the torques about C.
$\boldsymbol{\tau}$ = vector sum of all the torques about the origin O.
$\mathbf{F}$ = vector sum of all the external forces.

Theorem: $\boldsymbol{\tau} = \boldsymbol{\tau}_{\rm C} + \bar{\mathbf{r}} \times \mathbf{F}$. 3.8.1

Thus:
$$\boldsymbol{\tau} = \sum \mathbf{r}_i \times \mathbf{F}_i = \sum (\mathbf{r}'_i + \bar{\mathbf{r}}) \times \mathbf{F}_i$$
$$= \sum \mathbf{r}'_i \times \mathbf{F}_i + \bar{\mathbf{r}} \times \sum \mathbf{F}_i .$$

$$\therefore \qquad \boldsymbol{\tau} = \boldsymbol{\tau}_{\rm C} + \bar{\mathbf{r}} \times \mathbf{F}.$$

3.9 *Comparison*

At this stage I compare some somewhat similar formulas.

$$\mathbf{L} = \mathbf{L}_{\rm C} + \bar{\mathbf{r}} \times \mathbf{P} \qquad \boldsymbol{\tau} = \boldsymbol{\tau}_{\rm C} + \bar{\mathbf{r}} \times \mathbf{F}$$
$$\mathbf{L} = \sum m_i \mathbf{r}_i \times \mathbf{v}_i \qquad \boldsymbol{\tau} = \sum m_i \mathbf{r}_i \times \dot{\mathbf{v}}_i$$
$$\mathbf{L}_{\rm C} = \sum m_i \mathbf{r}'_i \times \mathbf{v}'_i \qquad \boldsymbol{\tau}' = \sum m_i \mathbf{r}'_i \times \dot{\mathbf{v}}_i$$
$$\mathbf{P} = \sum m_i \mathbf{v}_i \qquad \mathbf{F} = \sum m_i \dot{\mathbf{v}}_i$$

3.10 *Kinetic energy*

We remind ourselves that we are discussing *particles*, and that all kinetic energy is translational kinetic energy.

Notation: $T_{\rm C}$ = kinetic energy with respect to the centre of mass C.
T = kinetic energy with respect to the origin O.

Theorem: $T = T_{\rm C} + \frac{1}{2} M \bar{\upsilon}^2$. 3.10.1

Thus:
$$T = \tfrac{1}{2} \sum m_i \upsilon_i^2 = \tfrac{1}{2} \sum m_i (\mathbf{v}'_i + \bar{\mathbf{v}}) \bullet (\mathbf{v}'_i + \bar{\mathbf{v}})$$
$$= \tfrac{1}{2} \sum m_i \upsilon'^2_i + \bar{\mathbf{v}} \bullet \sum m_i \mathbf{v}'_i + \tfrac{1}{2} \bar{\upsilon}^2 \sum m_i .$$

$$\therefore \qquad T = T_{\rm C} + \tfrac{1}{2} M \bar{\upsilon}^2.$$

Corollary: If $\bar{\mathbf{v}} = 0$, $T = T_{\rm C}$. (Think about what this means.)

Corollary: For a non-rotating rigid body, $T_{\rm C} = 0$, and therefore $T = \frac{1}{2} M \bar{\upsilon}^2$.
(Think about what this means.)

3.11 *Torque and Rate of Change of Angular Momentum*

Theorem: *The rate of change of the total angular momentum of a system of particles is equal to the sum of the external torques on the system.*

Thus:

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i \tag{3.11.1}$$

∴

$$\dot{\mathbf{L}} = \sum_i \dot{\mathbf{r}}_i \times \mathbf{p}_i + \sum_i \mathbf{r}_i \times \dot{\mathbf{p}}_i . \tag{3.11.2}$$

But the first term is zero, because $\dot{\mathbf{r}}_i$ and $\mathbf{p}_i$ are parallel.

Also

$$\dot{\mathbf{p}}_i = \mathbf{F}_i + \sum_j \mathbf{F}_{ij} . \tag{3.11.3}$$

∴

$$\begin{aligned}\dot{\mathbf{L}} &= \sum_i \mathbf{r}_i \times \left(\mathbf{F}_i + \sum_j \mathbf{F}_{ij} \right) = \sum_i \mathbf{r}_i \times \mathbf{F}_i + \sum_i \mathbf{r}_i \times \sum_j \mathbf{F}_{ij} \\ &= \sum_i \mathbf{r}_i \times \mathbf{F}_i + \sum_i \sum_j \mathbf{r}_i \times \mathbf{F}_{ij} .\end{aligned}$$

But $\sum_i \sum_j \mathbf{F}_{ij} = 0$ by Newton's third law of motion, and so $\sum_i \sum_j \mathbf{r}_i \times \mathbf{F}_{ij}$ is also zero.

Also, $\sum_i \mathbf{r}_i \times \mathbf{F}_i = \boldsymbol{\tau}$, and so we arrive at

$$\dot{\mathbf{L}} = \boldsymbol{\tau} , \tag{3.11.4}$$

which was to be demonstrated.

Corollary: If the sum of the external torques on a system is zero, the angular momentum is constant. (Law of Conservation of Angular Momentum.)

3.12 *Torque, Angular Momentum and a Moving Point*

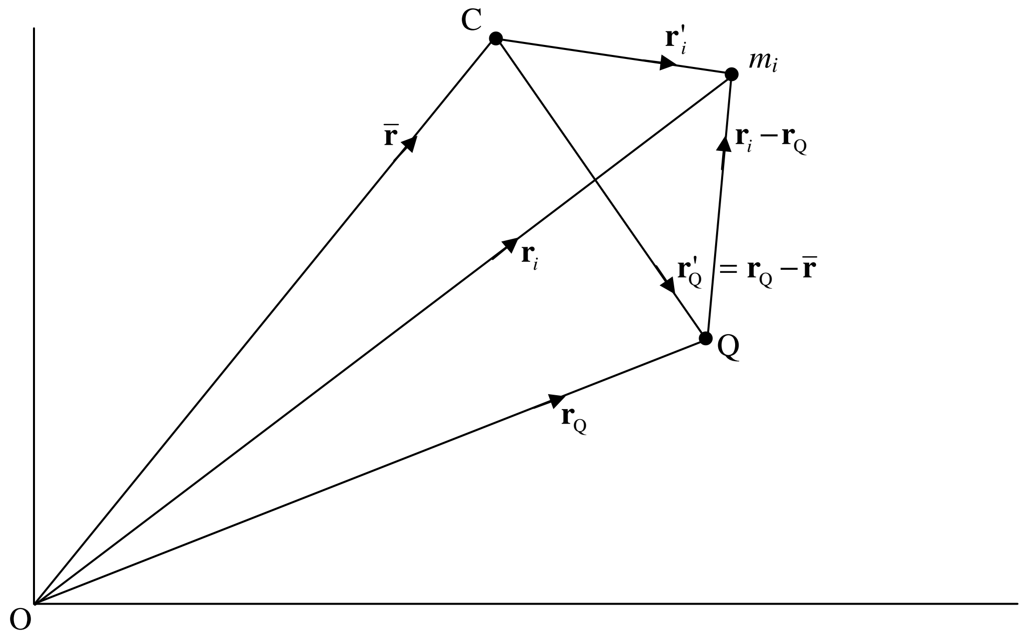

FIGURE III.7

In figure III.7 I draw the particle m_i, which is just one of n particles, $n - 1$ of which I haven't drawn and are scattered around in 3-space. I draw an arbitrary origin O, the centre of mass C of the system, and another point Q, which may (or may not) be moving with respect to O. The question I am going to ask is: Does the equation $\dot{\mathbf{L}} = \boldsymbol{\tau}$ apply to the point Q? It obviously does if Q is stationary, just as it applies to O. But what if Q is moving? If it does not apply, just what *is* the appropriate relation?

The theorem that we shall prove – and interpret – is

$$\dot{\mathbf{L}}_\text{Q} = \boldsymbol{\tau}_\text{Q} + M\mathbf{r}'_\text{Q} \times \ddot{\mathbf{r}}_\text{Q} . \qquad 3.12.1$$

We start: $$\mathbf{L}_\text{Q} = \sum \left(\mathbf{r}_i - \mathbf{r}_\text{Q}\right) \times \left[m_i\left(\mathbf{v}_i - \mathbf{v}_\text{Q}\right)\right]. \qquad 3.12.2$$

$$\therefore \qquad \dot{\mathbf{L}}_\text{Q} = \sum (\mathbf{r}_i - \mathbf{r}_\text{Q}) \times m_i(\dot{\mathbf{v}}_i - \dot{\mathbf{v}}_\text{Q}) + \sum (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_\text{Q}) \times m_i(\mathbf{v}_i - \mathbf{v}_\text{Q}) . \qquad 3.12.3$$

The second term is zero, because $\dot{\mathbf{r}} = \mathbf{v}$.
Continue:

$$\dot{\mathbf{L}}_{\mathrm{Q}} = \sum (\mathbf{r}_i - \mathbf{r}_{\mathrm{Q}}) \times m_i \dot{\mathbf{v}}_i - \sum m_i \mathbf{r}_i \times \dot{\mathbf{v}}_{\mathrm{Q}} + \sum m_i \mathbf{r}_{\mathrm{Q}} \times \dot{\mathbf{v}}_{\mathrm{Q}}. \qquad 3.12.4$$

Now $m_i \dot{\mathbf{v}}_i = \mathbf{F}_i$, so that the first term is just $\boldsymbol{\tau}_{\mathrm{Q}}$.

Continue:

$$\begin{aligned} \dot{\mathbf{L}}_{\mathrm{Q}} &= \boldsymbol{\tau}_{\mathrm{Q}} - \sum m_i \mathbf{r}_i \times \dot{\mathbf{v}}_{\mathrm{Q}} + M\mathbf{r}_{\mathrm{Q}} \times \dot{\mathbf{v}}_{\mathrm{Q}} \\ &= \boldsymbol{\tau}_{\mathrm{Q}} - M\overline{\mathbf{r}} \times \ddot{\mathbf{r}}_{\mathrm{Q}} + M\mathbf{r}_{\mathrm{Q}} \times \ddot{\mathbf{r}}_{\mathrm{Q}} \\ &= \boldsymbol{\tau}_{\mathrm{Q}} + M(\mathbf{r}_{\mathrm{Q}} - \overline{\mathbf{r}}) \times \ddot{\mathbf{r}}_{\mathrm{Q}}. \end{aligned}$$

$\therefore$ $$\dot{\mathbf{L}}_{\mathrm{Q}} = \boldsymbol{\tau}_{\mathrm{Q}} + M\mathbf{r}'_{\mathrm{Q}} \times \ddot{\mathbf{r}}_{\mathrm{Q}}. \qquad \text{Q.E.D.} \qquad 3.12.5$$

Thus in general, $\dot{\mathbf{L}}_{\mathrm{Q}} \neq \boldsymbol{\tau}_{\mathrm{Q}}$, but $\dot{\mathbf{L}}_{\mathrm{Q}} = \boldsymbol{\tau}_{\mathrm{Q}}$ under any of the following three circumstances:

i. $\mathbf{r}'_{\mathrm{Q}} = 0$ - that is, Q coincides with C.

ii. $\ddot{\mathbf{r}}_{\mathrm{Q}} = 0$ - that is, Q is not accelerating.

iii. $\ddot{\mathbf{r}}_{\mathrm{Q}}$ and $\mathbf{r}'_{\mathrm{Q}}$ are parallel, which would happen, for example, if O were a centre of attraction or repulsion and Q were accelerating towards or away from O.

3.13 *The Virial Theorem*

First, let me say that I am not sure how this theorem got its name, other than that my Latin dictionary tells me that *vis*, *viris* means *force*, and its plural form, *vires*, *virium* is generally translated as *strength.* The term was apparently introduced by Rudolph Clausius of thermodynamics fame. We do not use the word *strength* in any particular technical sense in classical mechanics, although we do talk about the *tensile strength* of a wire, which is the force that it can summon up before it snaps. We use the word *energy* to mean *the ability to do work*; perhaps we could use the word *strength* to mean *the ability to exert a force.* But enough of these idle speculations.

Before proceeding, I define the quantity

$$\mathcal{I} = \sum_i m_i r_i^2 \qquad 3.13.1$$

as the second moment of mass of a system of particles with respect to the origin. As discussed in Chapter 2, mass is (apart from some niceties in general relativity) synonymous with inertia, and the second moment of mass is used so often that it is nearly always called simply "the" moment of inertia, as though there were only one moment, the second, worth considering. Note carefully, however, that you are probably much more used to thinking about the moment of inertia with respect to an *axis* rather than with respect to a *point*. This distinction is discussed in Chapter 2, section 19. Note also that, since the symbol I tends to be heavily used in any discussion of moments of inertia, for moment of inertia with respect to a point I am using the symbol $\mathcal{J}$.

I can also write equation 3.13.1 as

$$\mathcal{J} = \sum_i m_i (\mathbf{r}_i \cdot \mathbf{r}_i) \qquad 3.13.2$$

Differentiate twice with respect to time:

$$\dot{\mathcal{J}} = 2\sum_i m_i (\mathbf{r}_i \cdot \dot{\mathbf{r}}_i), \qquad 3.13.3$$

and

$$\ddot{\mathcal{J}} = 2\sum_i m_i (\dot{r}_i^2 + \mathbf{r}_i \cdot \ddot{\mathbf{r}}_i) \qquad 3.13.4$$

or

$$\ddot{\mathcal{J}} = 4T + 2\sum_i \mathbf{r}_i \cdot m_i \ddot{\mathbf{r}}_i, \qquad 3.13.5$$

where T is the *kinetic energy* of the system of particles. The sums are understood to be over all particles - i.e. i from 1 to n.

$m_i \ddot{\mathbf{r}}_i$ is the force on the ith particle. I am now going to suppose that there are no *external* forces on any of the particles in the system, but the particles interact with each other with conservative forces, $\mathbf{F}_{ij}$ being the force exerted on particle i by particle j. I am also going to introduce the notation $\mathbf{r}_{ji} = \mathbf{r}_j - \mathbf{r}_i$, which is a vector directed from particle i to particle j. The relation between these three vectors in shown in figure III.8.

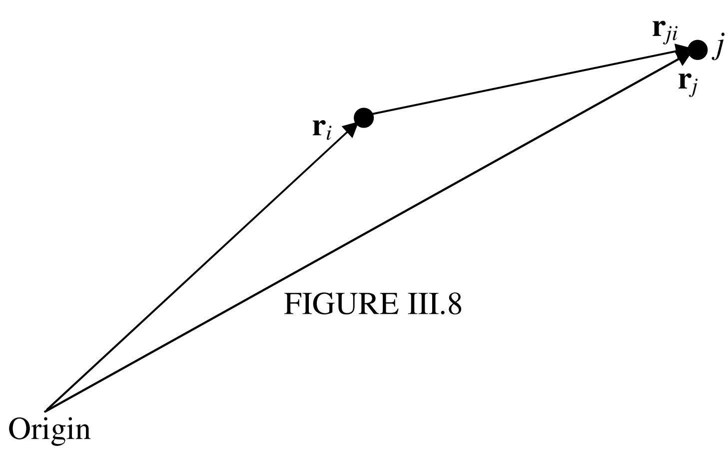

FIGURE III.8

I have not drawn the force $\mathbf{F}_{ij}$, but it will be in the opposite direction to $\mathbf{r}_{ji}$ if it is a repulsive force and in the same direction as $\mathbf{r}_{ji}$ if it is an attractive force.

The total force on particle i is $\sum_{j\neq i}\mathbf{F}_{ij}$, and this is equal to $m_i\ddot{\mathbf{r}}_i$. Therefore, equation 3.13.5 becomes

$$\ddot{\mathcal{J}} = 4T + 2\sum_i \mathbf{r}_i \cdot \sum_{j\neq i}\mathbf{F}_{ij}. \qquad 3.13.6$$

Now it is clear that

$$\sum_i \mathbf{r}_i \cdot \sum_{j\neq i}\mathbf{F}_{ij} = \sum_i \sum_{j<i} \mathbf{r}_{ij} \cdot \mathbf{F}_{ij}. \qquad 3.13.7$$

However, in case, like me, you find double subscripts and summations confusing and you have really no idea what equation 3.13.7 means, and it is by no means at all clear, I write it out in full in the case where there are five particles. Thus:

$$\begin{aligned}\sum_i \mathbf{r}_i \cdot \sum_{j\neq i}\mathbf{F}_{ij} = \quad & \mathbf{r}_1 \cdot (\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_{15}) \\ & + \mathbf{r}_2 \cdot (\mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \mathbf{F}_{25}) \\ & + \mathbf{r}_3 \cdot (\mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \mathbf{F}_{35}) \\ & + \mathbf{r}_4 \cdot (\mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43} + \mathbf{F}_{45}) \\ & + \mathbf{r}_5 \cdot (\mathbf{F}_{51} + \mathbf{F}_{52} + \mathbf{F}_{53} + \mathbf{F}_{54}).\end{aligned}$$

Now apply Newton's third law of motion:

$$\begin{aligned}\sum_i \mathbf{r}_i \cdot \sum_{j\neq i}\mathbf{F}_{ij} = \quad & \mathbf{r}_1 \cdot (-\mathbf{F}_{21} - \mathbf{F}_{31} - \mathbf{F}_{41} - \mathbf{F}_{51}) \\ & + \mathbf{r}_2 \cdot (\mathbf{F}_{21} - \mathbf{F}_{32} - \mathbf{F}_{42} - \mathbf{F}_{52}) \\ & + \mathbf{r}_3 \cdot (\mathbf{F}_{31} + \mathbf{F}_{32} - \mathbf{F}_{43} - \mathbf{F}_{53}) \\ & + \mathbf{r}_4 \cdot (\mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43} - \mathbf{F}_{54}) \\ & + \mathbf{r}_5 \cdot (\mathbf{F}_{51} + \mathbf{F}_{52} + \mathbf{F}_{53} + \mathbf{F}_{54}).\end{aligned}$$

Now bear in mind that $\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r}_{21}$, and we see that this becomes

$$\sum_i \mathbf{r}_i \cdot \sum_{j\neq i} \mathbf{F}_{ij} = \mathbf{F}_{21}\bullet\mathbf{r}_{21} + \mathbf{F}_{31}\bullet\mathbf{r}_{31} + \mathbf{F}_{41}\bullet\mathbf{r}_{41} + \mathbf{F}_{51}\bullet\mathbf{r}_{51}$$
$$+ \mathbf{F}_{32}\bullet\mathbf{r}_{32} + \mathbf{F}_{42}\bullet\mathbf{r}_{42} + \mathbf{F}_{52}\bullet\mathbf{r}_{52}$$
$$+ \mathbf{F}_{43}\bullet\mathbf{r}_{43} + \mathbf{F}_{53}\bullet\mathbf{r}_{53}$$
$$+ \mathbf{F}_{54}\bullet\mathbf{r}_{54}$$

and we have arrived at equation 3.13.7. Equation 3.13.6 then becomes

$$\ddot{\mathcal{J}} = 4T + 2\sum_i \sum_{j<i} \mathbf{r}_{ij} \cdot \mathbf{F}_{ij}. \qquad 3.13.8$$

This is the most general form of the virial equation. It tells us whether the cluster is going to disperse ($\ddot{\mathcal{J}}$ positive) or collapse ($\ddot{\mathcal{J}}$ negative) – though this will evidently depend on the nature of the force law $\mathbf{F}_{ij}$.

Now suppose that the particles attract each other with a force that is inversely proportional to the *n*th power of their distance apart. For gravitating particles, of course, $n = 2$. The force between two particles can then be written in various forms, such as

$$\mathbf{F}_{ij} = -F_{ij}\,\hat{\mathbf{r}}_{ij} = -\frac{k}{r_{ij}^n}\hat{\mathbf{r}}_{ij} = -\frac{k}{r_{ij}^{n+1}}\mathbf{r}_{ij}, \qquad 3.13.9$$

and the mutual potential energy between two particles is minus the integral of $F_{ij}dr$, or

$$U_{ij} = -\frac{k}{(n-1)r^{n-1}}. \qquad 3.13.10$$

I now suppose that the forces between the particles are gravitational forces, such that

$$\mathbf{F}_{ij} = -\frac{Gm_i m_j}{r_{ij}^3}\mathbf{r}_{ij}. \qquad 3.13.11$$

Now return to the term $\mathbf{r}_{ij} \bullet \mathbf{F}_{ij}$, which occurs in equation 3.13.8:

$$\mathbf{r}_{ij} \bullet \mathbf{F}_{ij} = -\frac{k}{r_{ij}^{n+1}}\mathbf{r}_{ij} \bullet \mathbf{r}_{ij} = -\frac{k}{r_{ij}^{n-1}} = (n-1)U_{ij}. \qquad 3.13.12$$

Thus equation 3.13.8 becomes

$$\ddot{\mathcal{I}} = 4T + 2(n-1)U, \qquad 3.13.13$$

where T and U are the kinetic and potential energies of the system. Note that for gravitational interaction (or any attractive) forces, the quantity U is *negative*. **Equation 3.13.13 is the virial theorem for a system of particles with an r^{-2} attractive force between them.** The system will disperse or collapse according the sign of $\ddot{\mathcal{I}}$. For a system of **gravitationally-interacting** particles, $n = 2$, and so the virial theorem takes the form

$$\ddot{\mathcal{I}} = 4T + 2U. \qquad 3.13.14$$

Of course, as the individual particles move around in the system, $\mathcal{I}$, T and U are all changing from moment to moment, but always in such a manner that equation 3.13.13 is satisfied.

In a *stable, bound* system, by which I mean that, over a long period of time, there is no long-term change in the moment of inertia of the system, and the system is neither irreversibly dispersing or contracting, that is to say in a system in which the average value of $\ddot{\mathcal{I}}$ over a long period of time is zero (I'll define "long" soon), the virial theorem for a stable, bound system of r^{-n} particles takes the form

$$2\langle T\rangle + (n-1)\langle U\rangle = 0, \qquad 3.13.15$$

and **for a <u>stable</u> system of gravitationally-interacting particles,**

$$2\langle T\rangle + \langle U\rangle = 0, \qquad 3.13.16$$

Here the angular brackets are understood to mean the average values of the kinetic and potential energies over a long period of time. By a "long" period we mean, for example, long compared with the time that a particle takes to cross from one side of the system to the other, or long compared with the time that a particle takes to move in an orbit around the centre of mass of the system. (In the absence of external forces, of course, the centre of mass does not move, or it moves with a constant velocity.)

For example, if a bound cluster of stars occupies a spherical volume of uniform density, the potential energy is $-\frac{3GM^2}{5a}$ (see equation 5.9.1 of Celestial Mechanics), so the virial theorem (equation 3.13.16) will enable you to work out the mean kinetic energy and hence speed of the stars. A globular cluster has roughly spherical symmetry, but it is not of uniform density, being centrally condensed. If you assume some functional form for the density distribution, this will give a slightly different formula for the potential energy, and you can then still use the virial theorem to calculate the mean kinetic energy.

A trivial example is to consider a planet of mass m moving in a circular orbit of radius a around a Sun of mass M, such that $m \ll M$ and the Sun does not move. The potential energy of the system is $U = -GMm/a$. The speed of the planet is given by equating $\frac{m\upsilon^2}{a}$ to $\frac{GMm}{a^2}$, from which $T = GMm/(2a)$, so we easily see in this case that $2T + U = 0$.