CHAPTER 16
HYDROSTATICS

16.1 *Introduction*

This relatively short chapter deals with the pressure under the surface of an incompressible fluid, which in practice means a liquid, which, compared with a gas, is nearly, if not quite, incompressible. It also deals with Archimedes' principle and the equilibrium of floating bodies. The chapter is perhaps a little less demanding than some of the other chapters, though it will assume a familiarity with the concepts of centroids and radius of gyration, which are dealt with in chapters 1 and 2.

16.2 *Density*

There is little to be said about density other than to define it as mass per unit volume. However, this expression does not literally mean the mass of a cubic metre, for after all a cubic metre is a large volume, and the density may well vary from point to point throughout the volume. Density is an intensive quantity in the thermodynamical sense, and is defined at every *point*. A more exact definition of density, for which I shall usually use the symbol ρ, is

$$\rho = \lim_{\delta V \to 0} \frac{\delta m}{\delta V} . \qquad 16.2.1$$

The awful term "specific gravity" was formerly used, and is still regrettably often heard, as either a synonym for density, or the dimensionless ratio of the density of a substance to the density of water. It should be avoided. The only concession I shall make is that I shall use the symbol *s* to mean the ratio of the density of a body to the density of a fluid in which it may be immersed or on which it may be floating,

The density of water varies with temperature, but at 4 ºC is 1 g cm^{-3} or 1000 kg m^{-3}, or 10 lb gal^{-1}. The original gallon was the volume of 10 pounds (lb) or water. These are Imperial (UK) gallons, and avoirdupois pounds - not the gallons (wet or dry) used in the U.S., and not the pounds (troy) used in the jewellery trade.

16.3 *Pressure*

Pressure is force per unit area, or, more precisely,

$$P = \lim_{\delta A \to 0} \frac{\delta F}{\delta A} . \qquad 16.3.1$$

There is no particular direction associated with pressure – it acts in all directions – and it is a scalar quantity. The SI unit is the *pascal* (Pa), which is a pressure of one newton per square metre (N m^{-2}). Blaise Pascal (1623-1662) was a French mathematician and philosopher who contributed greatly to the theory of conic sections and to hydrostatics. He showed that the barometric pressure decreases with height – hence the famous examination question: "Explain how you would use a barometer to measure the height of a tall building" – to which the most accurate answer is said to be: "I would drop it out of the window and time how long it took to reach the ground."

The CGS unit of pressure is dyne cm^{-2}, and 1 Pa = 10 dyne cm^{-2}.

Some other silly units for pressure are often seen, such as psi, bar, Torr or mm Hg, and atm.

A psi or "pound per square inch" is all right for those who define a "pound" as a unit of force (US usage) but is less so for those who define a pound as a unit of mass (UK usage). A psi is about 6894.76 Pa.

[The "British Engineering System", as far as I know, is used exclusively in the U.S. and is not and never has been used in Britain, where it would probably be unrecognized. In the "British" Engineering System, the pound is defined as a unit of force, whereas in Britain a pound is a unit of mass.]

A bar is 10^5 Pa or 100 kPa.

A Torr is a pressure under a column of mercury 760 mm high. This may be convenient for casual conversational use where extreme precision is not expected in laboratory experiments in which pressure is actually indicated by a mercury barometer or manometer. To find out exactly what the pressure in Pa under 760 mm Hg is, one would have to know the exact value of the local gravitational acceleration and also the exact density of mercury, which varies with temperature and with isotopic constitution. A Torr is usually given as 133.322 Pa. Evangelista Torricelli (1608 – 1647) is regarded as the inventor of the mercury barometer. He succeeded Galileo as professor of mathematics at the University of Florence.

An atm is 760 torr or about 14.7 psi or 101 325 Pa. That is to say, 1.013 25 bar

As usual, the use of a variety of different units, and knowing the exact definitions and conversion factors between all of them and carrying out all the tedious multiplications, is an unnecessary chore that is inflicted upon all of us in all branches of physics.

16.4 *Pressure on a Horizontal Surface. Pressure at Depth z*

FIGURE XVI.1

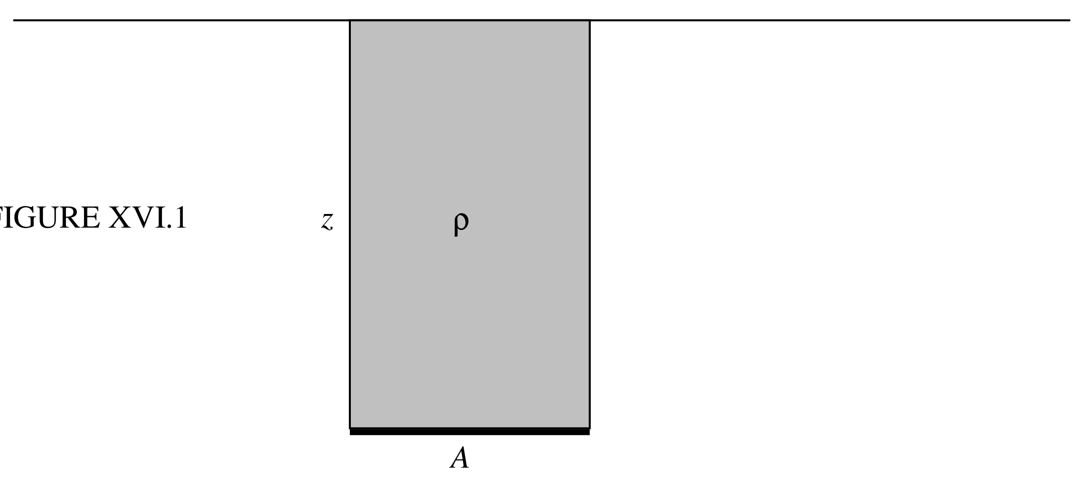


Figure XVI.1 shows a horizontal surface of area A immersed at a depth z under the surface of a fluid at depth z. The force F on the area A is equal to the weight of the superincumbent fluid. This gives us occasion to use a gloriously pompous word. "Incumbent" means "lying down", so that "superincumbent" is lying down above the area. It is incumbent upon all of us to understand this. The weight of the superincumbent fluid is evidently its volume Az times its density ρ times the gravitational acceleration g. Thus

$$F = \rho gzA, \qquad 16.4.1$$

and, since pressure is force per unit area, we find that the pressure at a depth z is

$$P = \rho gz. \qquad 16.4.2$$

This is, of course, in addition to the atmospheric pressure that may exist above the surface of the liquid.

The pressure is the same at all points at the same horizontal level within a homogeneous incompressible fluid. This seemingly trivial statement may sometimes be worth remembering under the stress of examination conditions. Thus, let's look at an example.

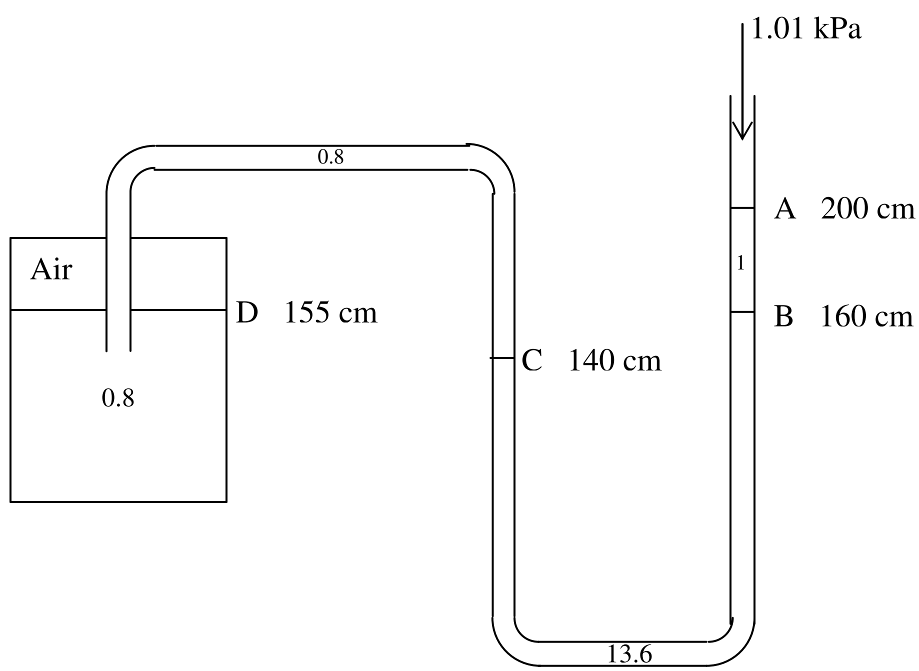


FIGURE XVI.2

In figure XVI.2, the vessel at the left is partly filled with a liquid of density 0.8 g cm^{-3}, the upper part of the vessel being filled with air. The liquid also fills the tube along to the point C. From C to B, the tube is filled with mercury of density 13.6 g cm^{-3}. Above that, from B to A, is water of density 1.0 g cm^{-3}, and above that is the atmospheric pressure of 101 kPa. The height of the four interfaces above the thick black line are

A	200 cm
B	160 cm
C	140 cm
D	155 cm

With $g = 9.8$ m s^{-2}, what is the pressure of the air in the closed vessel?

I'll do the calculation in SI units.

Pressure at A = 101000 Pa

Pressure at B = $101000 + 1000 \times 9.8 \times 0.4 = 104920$ Pa

Pressure at C = $104920 + 13600 \times 9.8 \times 0.20 = 131576$ Pa

Pressure at D = $131576 - 800 \times 9.8 \times 0.15 = 130400$ Pa,

and this is the pressure of the air in the vessel.

Rather boring so far, and the next problem will also be boring, but the problem after that should keep you occupied arguing about it over lunch.

Problem. This problem is purely geometrical and nothing to do with hydrostatics – but the result will help you with the next problem after this. If you don't want to do it, just use the result in the next problem.

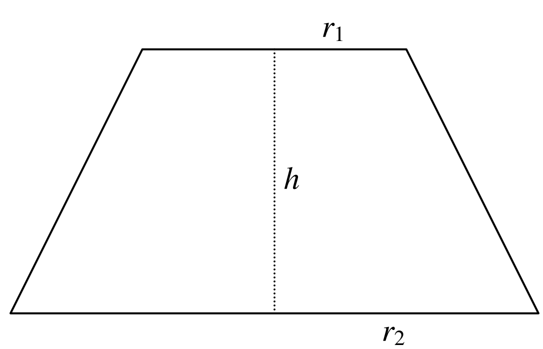


FIGURE XV1.3

Show that the volume of the frustrum of a cone, whose upper and lower circular faces are of radii r_1 and r_2, and whose height is h, is $\frac{1}{3}\pi h(r_1^2 + r_1 r_2 + r_2^2)$.

Problem. (Pascal's Paradox)

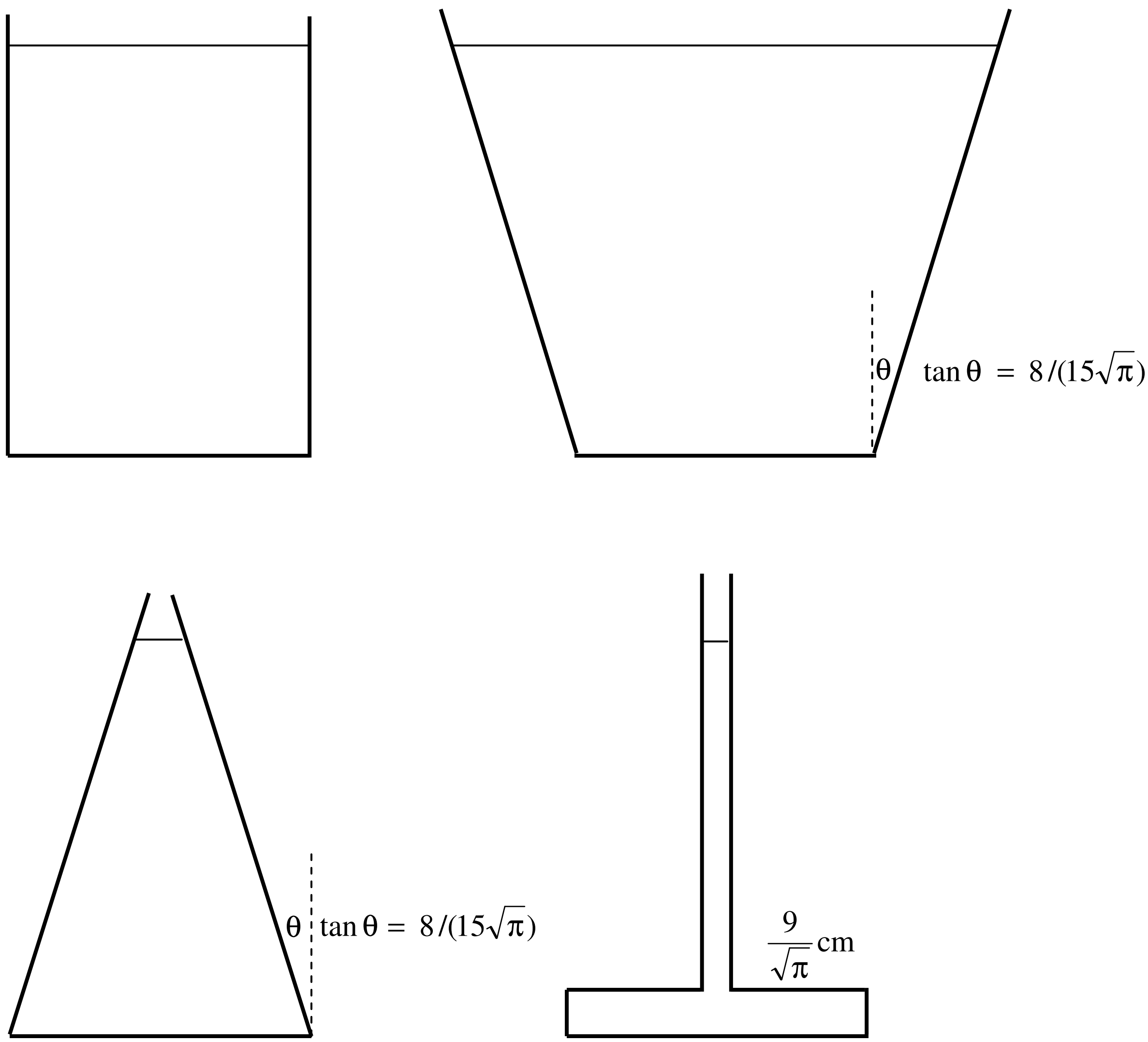


FIGURE XVI.4

Figure XVI.4 shows four vessels. The base of each is circular with the same radius, $10/\sqrt{\pi}$ cm, so the area is 100 cm^2. Each is filled with water (density = 1 g cm^{-3}) to a depth of 15 cm.

Calculate
1. The mass of water in each.
2. The pressure at the bottom of each vessel.
3. The force on the bottom of each vessel.

If the bottom of each vessel were made of glass, which cracked under a certain pressure, which would crack first if the vessels were slowly filled up? If the bottom of each vessel were welded to the scale of a weighing machine, what weight would be recorded?

I'll leave you to argue about this for as long as you wish.

16.5 *Pressure on a Vertical Surface*

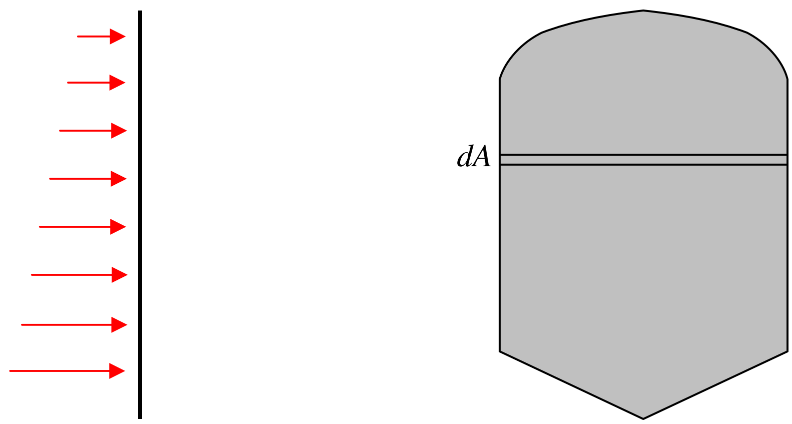


FIGURE XVI.5

Figure XVI.5 shows a vertical surface from the side and face-on. The pressure increases at greater depths. I show a strip of the surface at depth z. Suppose the area of that strip is dA. The pressure at depth z is ρgz, so the force on the strip is $\rho gzdA$. The force on the entire area is $\rho g\int z\,dA$, and that, by definition of the centroid (see chapter 1), is $\rho g\bar{z}A$, where $\bar{z}$ is the depth of the centroid. The same result will be obtained for an inclined surface. Therefore:

The total force on a submerged vertical or inclined plane surface is equal to the area of the surface times the depth of the centroid.

Example. Figure XVI.6 shows a triangular area. The uppermost side of the triangle is parallel to the surface at a depth z. The depth of the centroid is $z + \frac{1}{3}h$, so the pressure at the centroid is $\rho g(z + \frac{1}{3}h)$. The area of the triangle is $\frac{1}{2}bh$, so the total force on the triangle is $\frac{1}{2}\rho gbh(z + \frac{1}{3}h)$.

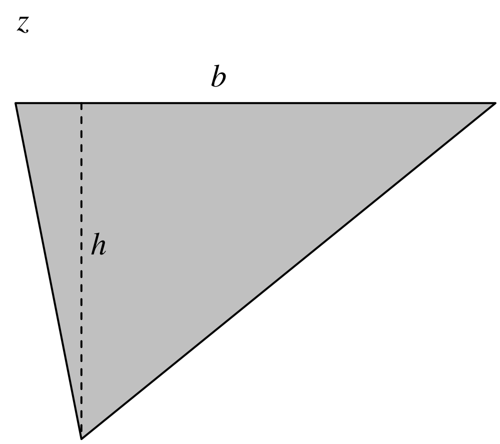


FIGURE XVI.6

16.6 *Centre of Pressure*

"The centre of pressure is the point at which the pressure may be considered to act." This is a fairly meaningless sentence, yet it is not entirely devoid of all meaning. If you refer to the left hand side of figure XVI.5, you will see an infinite number (I've drawn only eight) of forces. If you were to replace all of these forces by a single force, where would you put it? Or, more precisely, if you were to replace all of these forces by a single force such that the (first) *moment of this force* about a line through the surface of the fluid is the same as the (first) *moment of all the actual forces*, where would you place this single force? You would place it at the *centre of pressure.* The depth of the centre of pressure is a depth such that the moment of the total force on a vertical surface about a line in the surface of the fluid is the same as the moment of all the hydrostatic forces about a line in the surface of the fluid. I shall use the Greek letter ζ to indicate the depth of the centre of pressure. We can continue to use figure XVI.5.

The force on the strip of area dA at depth z is, as we have seen, $\rho gz dA$, so the first moment of that force is $\rho gz^2 dA$. The total moment is therefore $\rho g \int z^2 dA$, which is, by definition of radius of gyration k, (see chapter 2), $\rho g k^2 A$. The total force, as we have seen, is $\rho g \bar{z} A$, and the total moment is to be this times ζ. Thus the depth of the centre of pressure is

$$\zeta = \frac{k^2}{\bar{z}}. \qquad 16.6.1$$

Example.

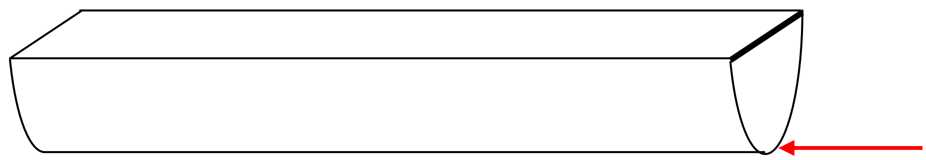


FIGURE XVI.7

A semicircular trough of radius a is filled with water, density ρ. One semicircular end of the trough is freely hinged at its diameter (the thick line in the figure). What force must be exerted at the bottom of the trough to prevent the end from swinging open?

The area of the semicircle is $\frac{1}{2}\pi a^2$. The depth of the centroid is $\frac{4a}{3\pi}$ so the total hydrostatic force is $\frac{2}{3}\rho g a^2$. The square of the radius of gyration is $\frac{1}{4}a^2$, so the depth of the centre of pressure is $\zeta = \frac{3\pi}{16}a$. The moment of the hydrostatic forces is therefore $\frac{1}{8}\pi\rho g a^3$. If the required force is F, this must equal Fa, and therefore $F = \frac{1}{8}\pi\rho g a^2$.

16.7 *Archimedes' Principle*

The most important thing about Archimedes' principle is to get the apostrophe in the right place and to spell principle correctly.

Archimedes was a Greek scientist who lived in Syracuse, Sicily. He was born about 287 BC and died about 212 BC. He made many contributions to mechanics. He invented the Archimedean screw, he is reputed to have said "Give me a fulcrum and I shall move the world", and he probably did not set the Roman invading fleet on fire by focussing sunlight on them with concave mirrors – though it makes a good story. The most famous story about him is that he was commissioned by King Hiero of Sicily to determine whether the king's crown was contaminated with base metal. Archimedes realized that he would need to know the density of the crown. Measuring its weight was no problem, but – how to measure the volume of such an irregularly-shaped object? One day, he went to take a bath, and he had filled the bath full right to the rim. When he stepped into the bath he was much surprised that some of the water slopped over the edge of the bath on to the floor. Suddenly, he realized that he had the solution to his problem, so straightway he raced out of the house and ran absolutely starkers through the streets of Syracuse shouting "Ευρηκα! Ευρηκα!", which is Greek for:

When a body is totally or partially immersed in a fluid, it experiences a hydrostatic upthrust equal to the **weight** of fluid displaced.

Figure XVI.8 is a drawing of some water or other fluid. I have outlined with a dashed curve an arbitrary portion of the fluid. It is subject to hydrostatic pressure from the rest of the fluid. The small pressure of the fluid above it is pushing it down; the larger pressure of the fluid below it is pushing it up. Therefore there is a net upthrust. The portion of the fluid outlined is in equilibrium between its own weight and the hydrostatic upthrust. If we were to replace this portion of the fluid with a lump of iron, we wouldn't have changed the hydrostatic forces. Therefore the upthrust is equal to the **weight** of fluid displaced.

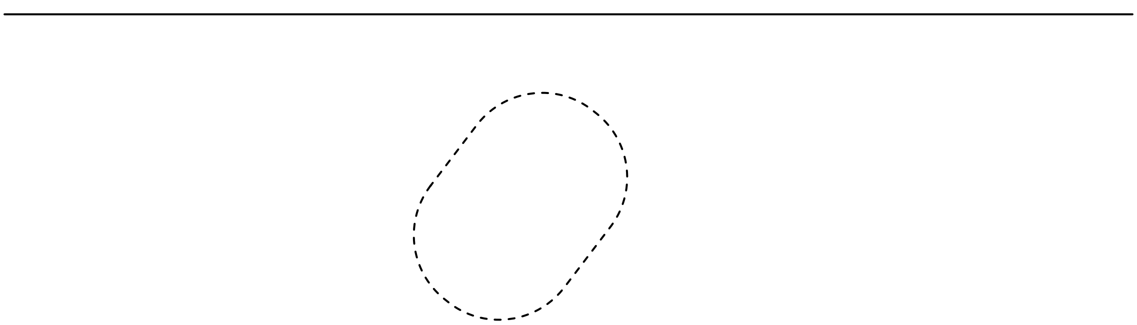


FIGURE XVI.8

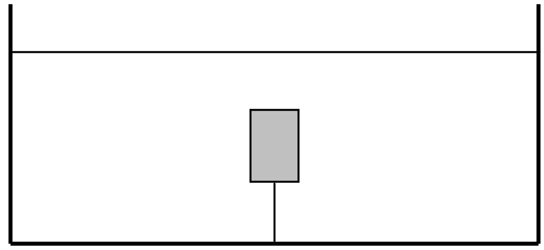
If a body is *floating* on the surface, the hydrostatic upthrust, as well as being equal to the **weight** of fluid displaced, is also equal to the weight of the body.

16.8 *Some Simple Examples*

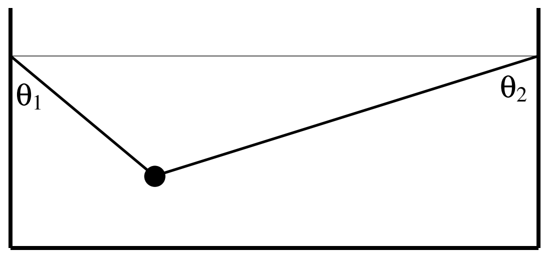
As we pointed out in the introduction to this chapter, this chapter is less demanding than some of the others, and indeed it has been quite trivial so far. Just to show how easy the topic is, here are a few quick examples.

1. A cylindrical vessel of cross-sectional area A is partially filled with water. A mass m of ice floats on the surface. The density of water is ρ_0 and the density of ice is ρ. Calculate the change in the level of the water when the ice melts, and state whether the water level rises or falls.

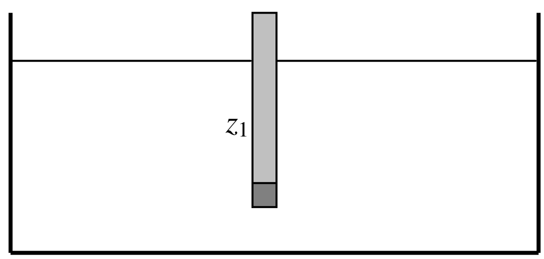
2. A cork of mass m, density ρ, is held under water (density ρ_0) by a string. Calculate the tension in the string. Calculate the initial acceleration if the string is cut.



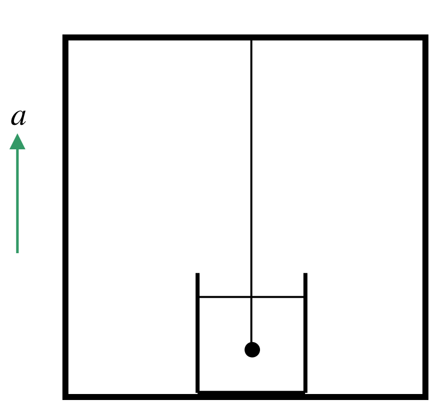
3. A lump of lead (mass m, density ρ) is held hanging in water (density ρ_0) by two strings as shown. Calculate the tension in the strings.



4. A hydrometer (for our purposes a hydrometer is a wooden rod weighted at the bottom for stability when it floats vertically) floats in equilibrium to a depth z_1 in water of density ρ_1. If salt is added to the water so that the new density is ρ_2, what is the new depth z_2?

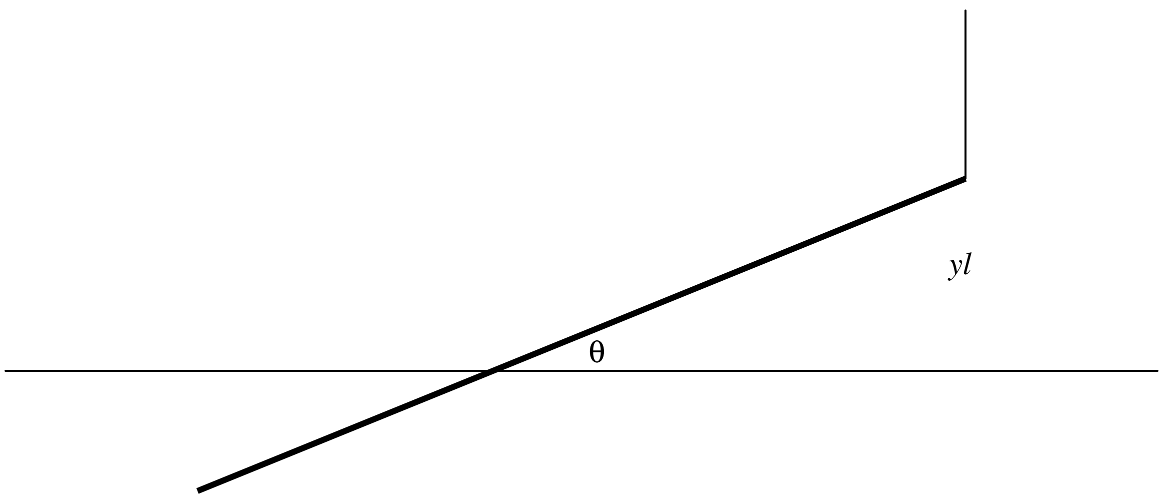


5. A mass m, density ρ, hangs in a fluid of density ρ_0 from the ceiling of an elevator (lift). The elevator accelerates upwards at a rate a. Calculate the tension in the string.



6. A hydrometer of mass m and cross-sectional area A floats in equilibrium do a depth h in a liquid of density ρ. The hydrometer is then gently pushed down and released. Determine the period of oscillation.

7. A rod of length l and density $s\rho$ ($s < 1$) floats in a liquid of density ρ. One end of the rod is lifted up through a height yl so that a length xl remains immersed. I have drawn it with the rope vertical. Must it be?)



i. Find x as a function of s.

ii. Find θ as a function of y and s.

iii. Find the tension T in the rope as a function of m, g and s.

Draw the following graphs:

a. x and $T/(mg)$ versus s.

b. θ versus y for several s.

c. θ versus s for several y.

d. x versus y for several s.

e. $T/(mg)$ versus y for several s.

Answers

1. No, it doesn't.

2. $$T = \left(\frac{\rho_0 - \rho}{\rho}\right) mg$$

3. $$T_1 = \frac{\left(\frac{\rho - \rho_0}{\rho}\right) mg}{\cos\theta_1 + \frac{\sin\theta_1}{\tan\theta_2}} \qquad T_2 = \frac{\left(\frac{\rho - \rho_0}{\rho}\right) mg}{\cos\theta_2 + \frac{\sin\theta_2}{\tan\theta_1}}$$

4. $$z_2 = \frac{\rho_1}{\rho_2} z_1$$

5. $$T = m\left[a + g\left(\frac{\rho - \rho_0}{\rho}\right)\right]$$

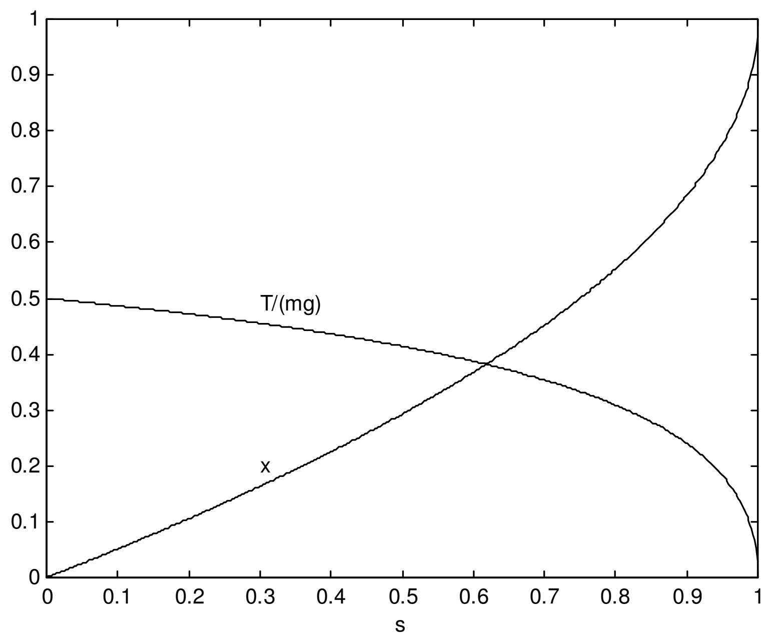
6. $$P = 2\pi\sqrt{\frac{m}{\rho A g}}$$

7. i. $$x = 1 - \sqrt{1 - s}$$

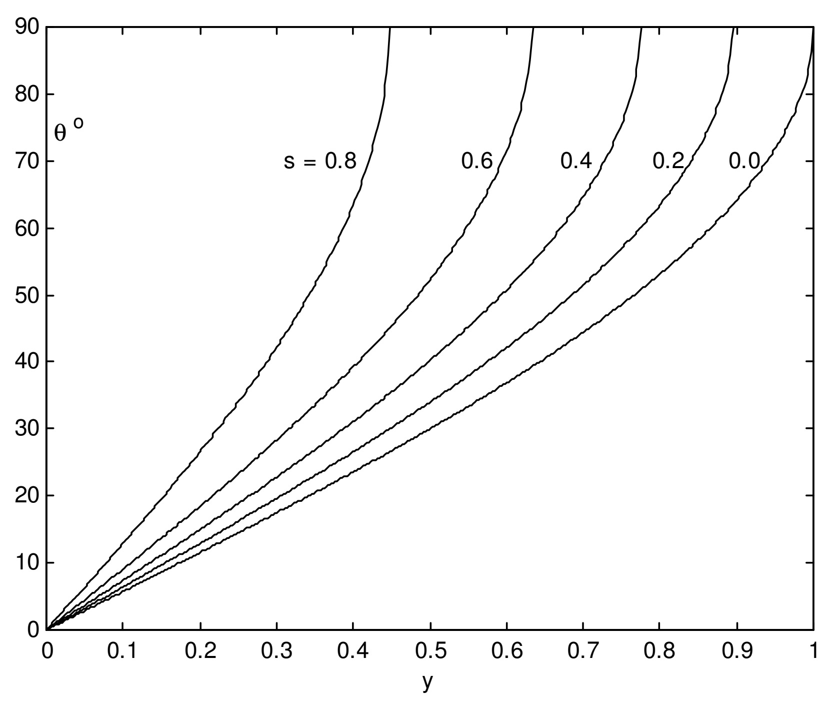
   ii. $$\sin\theta = \frac{y}{\sqrt{1 - s}}$$

   iii. $$T = mg\left(\frac{\sqrt{1 - s} - (1 - s)}{s}\right)$$

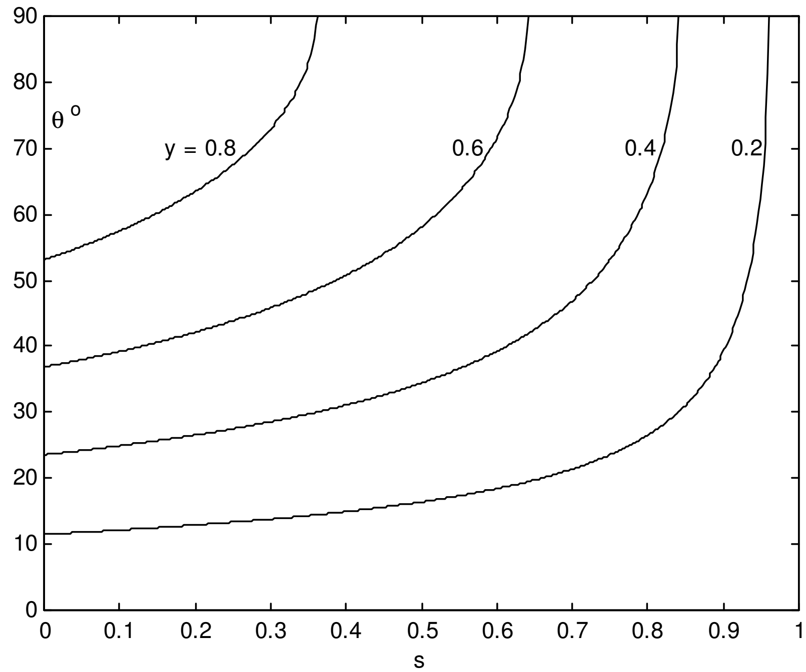
a.


T/(mg)
x
s
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

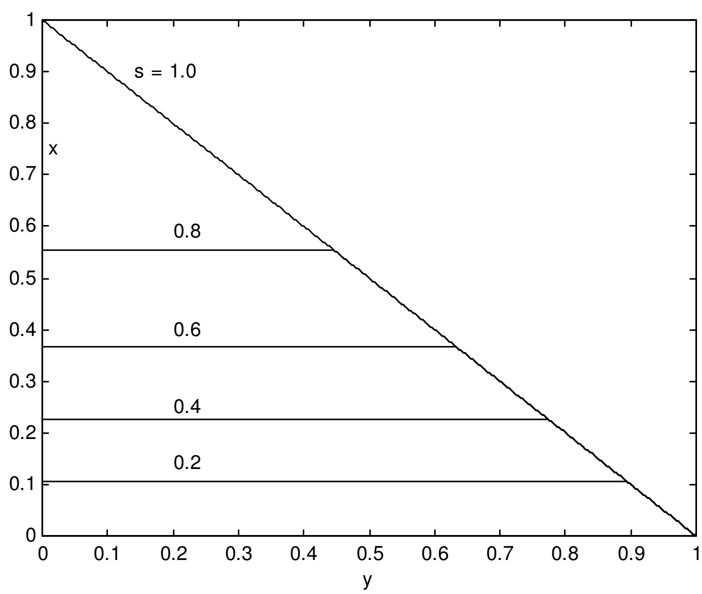
b.


θ °
s = 0.8
0.6
0.4
0.2
0.0
y
0
10
20
30
40
50
60
70
80
90
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

c.


θ °
y = 0.8
0.6
0.4
0.2
90
80
70
60
50
40
30
20
10
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
s

d.


s = 1.0
0.8
0.6
0.4
0.2
x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
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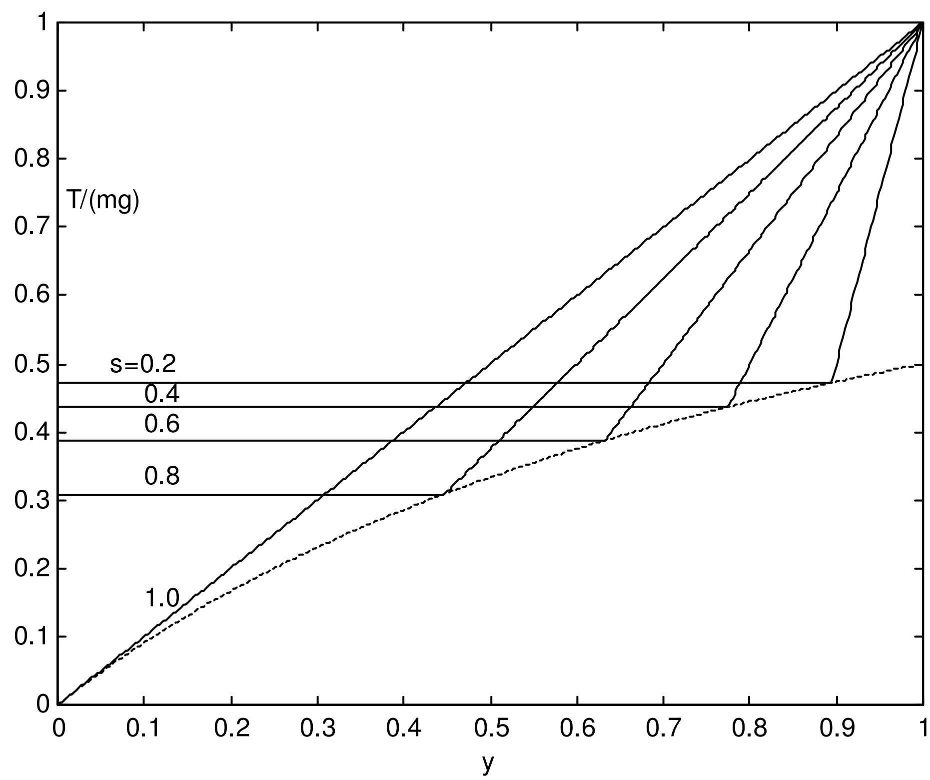
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
y

e.



16.9 *Floating Bodies*

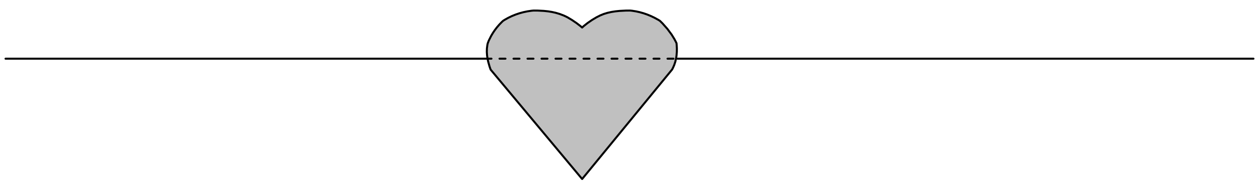
This is the most grisly topic in hydrostatics.

We can start with an observation that we have already made in section 16,7, namely that, if a body is freely floating, the hydrostatic upthrust is equal to the weight of the body.

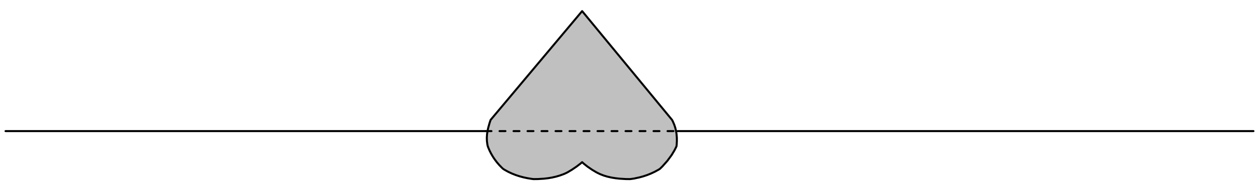
I also introduce here the term *centre of buoyancy*, which is the centre of mass of the displaced fluid. In a freely-floating body in equilibrium, the centre of buoyancy is vertically below the centre of mass of the floating body. As far as calculating the *moment* about some axis of the hydrostatic upthrust is concerned, the upthrust can be considered to act through the centre of buoyancy, just as the weight of an object can be considered to act through its centre of mass. See section 1.1 of chapter 1, for example, for a discussion of this point.

Also, before we get going, here is another small problem.

Problem 8. The drawing shows a body, whose relative density (i.e. its density relative to that of the fluid that it is floating in) is s_1. The dashed line is the *water-line section*.



Now, in the next drawing, a body of exactly the same size and shape (but not necessarily the same density) is floating upside down, with the same water-line section.



What is the relative density of this second body?
(Solution at the end of the chapter.)

I want to look now at the *stability* of equilibrium of a freely-floating body. While at first sight this may not be a very interesting topic, if you ever happen to be a passenger on an ocean liner, you might then find it to be quite interesting, for you will be interested to know, if the liner is given a small angular displacement from the vertical position, whether it will capsize and throw you into the sea, or whether it will right itself. Under such circumstances it becomes a very interesting subject indeed.

Before I start, I just want to establish one small geometric result.

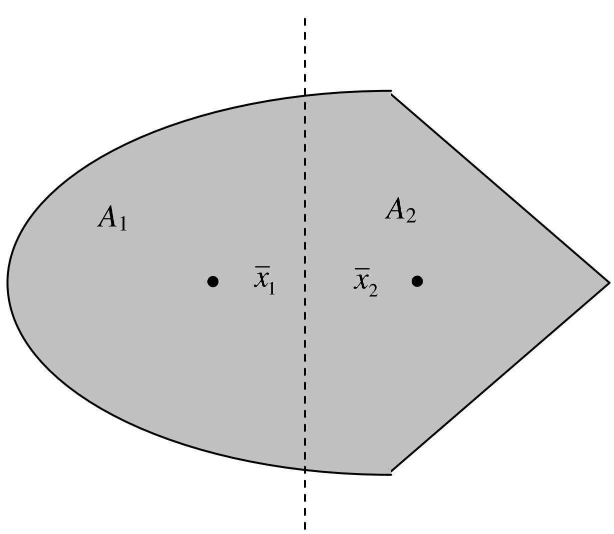


FIGURE XVI.9

Figure XVI.9 shows a plane bilaterally-symmetric area. I have drawn a dashed line through the centroid of the area. The areas to the left and right of this line are A_1 and A_2, and I have indicated the positions of the centroids of these two areas. (I haven't calculated the positions of the three centroids accurately – I just drew them approximately where I thought they would be.) Note that, since the dashed line goes through the centroid of the whole area, $A_1\bar{x}_1 = A_2\bar{x}_2$. Now rotate the area about the dashed line through an angle θ. By the theorem of Pappus (see chapter 1, section 1.6), the volume swept out by A_1 is $A_1 \times \bar{x}_1\theta$ and the volume swept out by A_2 is $A_2 \times \bar{x}_2\theta$. Thus we have established the geometrical result that I wanted, namely, that when a bilaterally symmetric area is rotated about an axis perpendicular to its axis of symmetry and passing through its centroid, the areas to left and right of the axis of rotation sweep out equal volumes.

We can now return to floating bodies, and I am going to consider the stability of equilibrium of a bilaterally symmetric floating body to a rotational displacement about an axis lying in the water line section and perpendicular to the axis of symmetry.

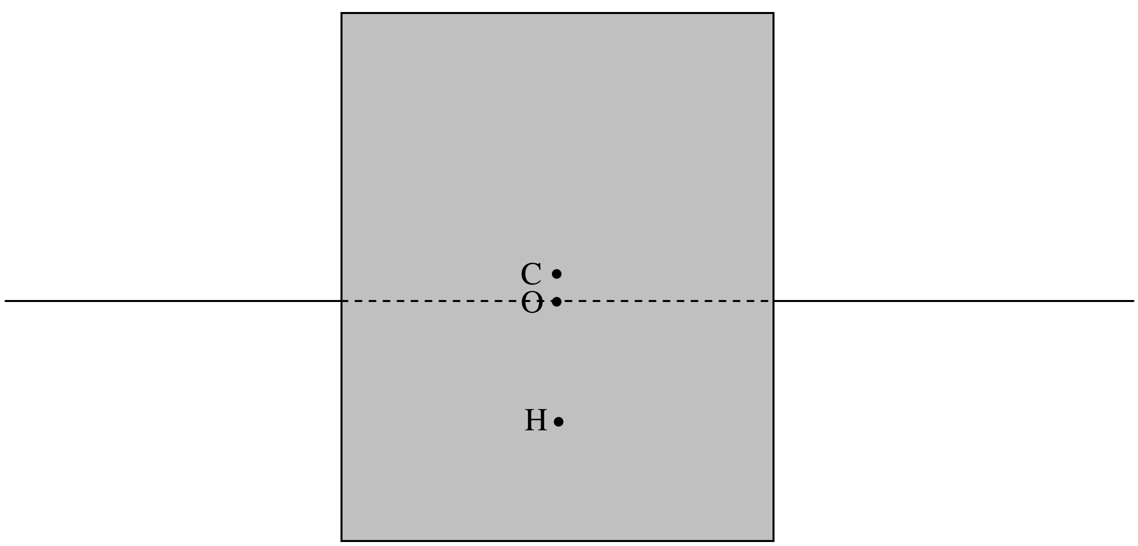


FIGURE XVI.10

I have drawn in figure XVI.10 the centre of mass C of the whole body, the centre of buoyancy H, and the centroid of the water-line section. The body is bilaterally symmetric about the plane of the paper, and we are going to rotate the body about an axis through O perpendicular to the plane of the paper, and we want to know whether the equilibrium is stable against such an angular displacement. We are going to rotate it in such a manner that the volume submerged is unaltered by the rotation – which means that the hydrostatic upthrust will remain equal to the weight of the body, and there will be no vertical acceleration. The geometrical theorem that we have just established shows that, if we rotate the body about an axis through the centroid of the water-line section, the volume submerged will be constant; conversely, our condition that the volume submerged is constant implies that the rotation is about an axis through the centroid of the water-line section.

I am going to establish a set of rectangular axes, origin O, with the *x*-axis to the right, the *y*-axis towards you, and the *z*-axis downwards. I'm going to call the depth of the centre H of buoyancy $\overline{z}$. Now let's carry out the rotation about O through an angle θ.

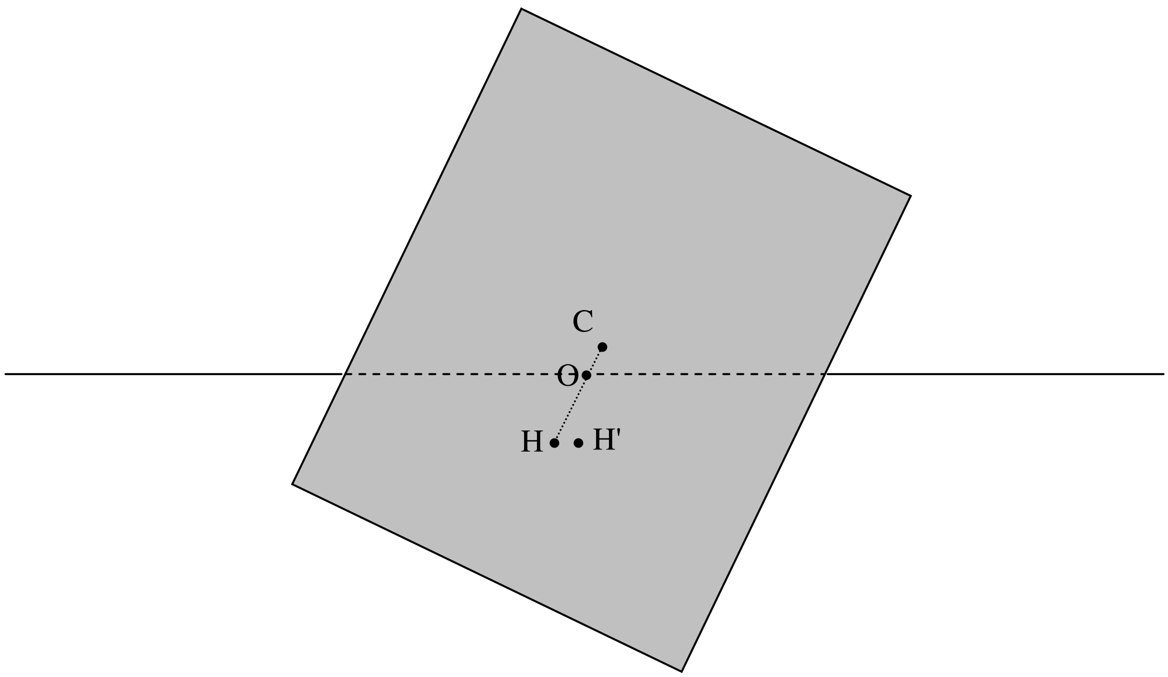


FIGURE XVI.11

I have drawn the position of the new centre of buoyancy H' and I wish to find its coordinates ($\overline{x}'$, $\overline{z}'$) relative to O. We shall find that it has moved a little horizontally compared with the original position of H, but its depth is almost unchanged. Indeed, for small θ, we shall find that $\overline{z}' - \overline{z}$ is of order θ^2, while $\overline{x}' - \overline{x}$ is of order θ. Thus, to first order in θ, I shall assume that the depth of the centre of buoyancy has remained unchanged.

However, the coordinate $\overline{x}'$ of the new centre of buoyancy will be of interest for the following reason. The weight of the body acts at its centre of mass C while the hydrostatic upthrust acts at the new centre of buoyancy H' and these two forces form a couple and exert a torque. You will understand from figure XVI.11 that if H' is to the left of C, the torque will topple the body over, whereas if H' is to the right of C, the torque will stabilize the body. Indeed, the horizontal distance between C and H' is known as the *righting lever*. The point on the line COH vertically above H' is called the *metacentre*. I haven't drawn it on the diagram, in order to minimise clutter, but I shall use the symbol M to indicate the metacentre. We can see that the condition for stability of equilibrium is that HM > HC. This is why we are interested in finding the exact position of the new centre of buoyancy H'.

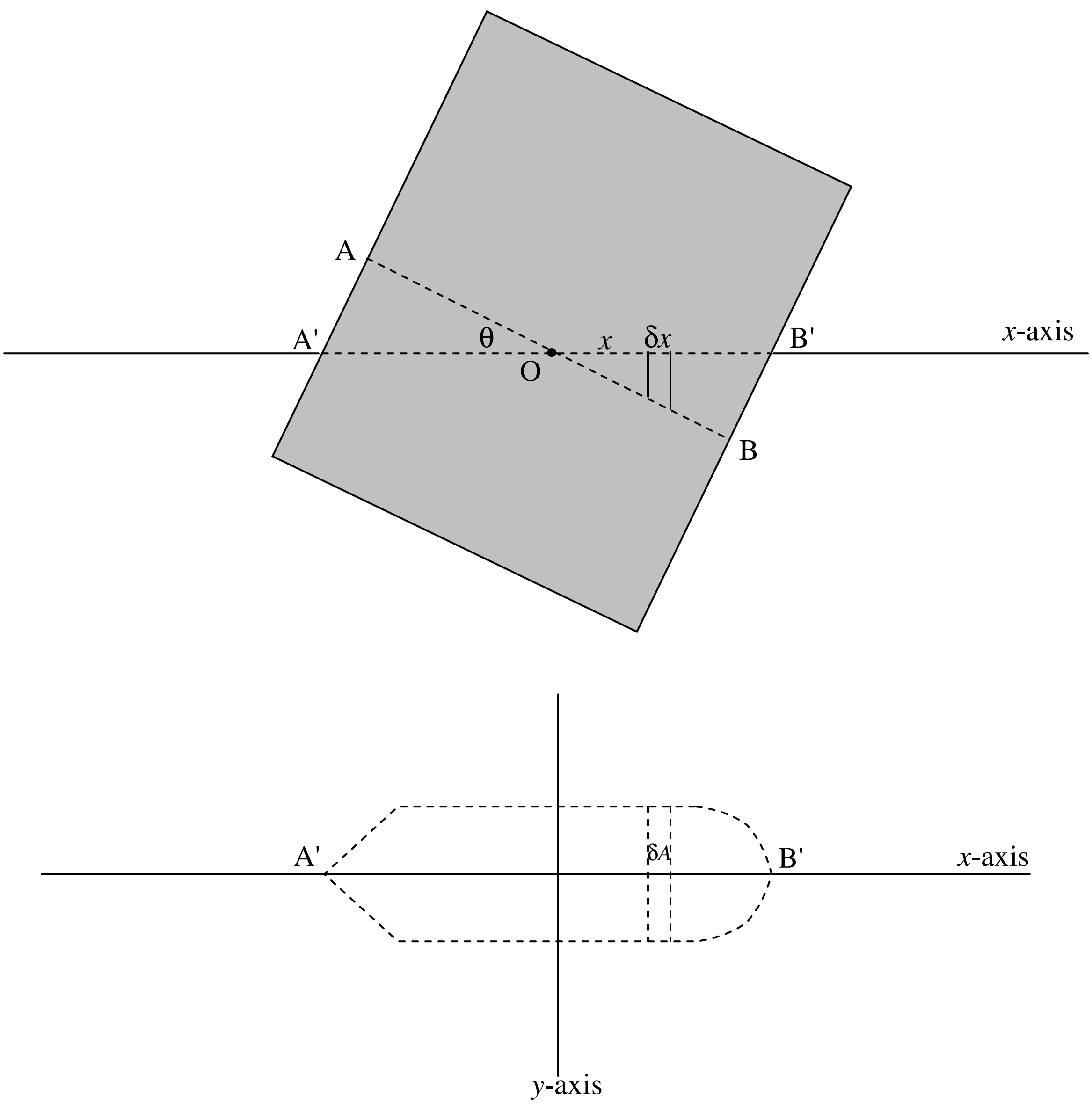


FIGURE XVI.12

In the upper part of figure XVI.12 I have drawn the old and new water-line sections as seen from the side, and in the lower part I have drawn the new water-line section seen from above. I have indicated an elemental volume of width δx of the displaced fluid at a distance x from the centroid O of the water-line section. For small θ the depth of this element is $x\theta$. Let's call its area in the water-line section δA, so that the volume element is $x\theta\delta A$. We'll call the total volume of the displaced fluid (which is unaltered by the rotation) V.

Consider the moments of volume about the x-axis. We have

$$V\,\bar{z}' = V\,\bar{z} - \int_{\mathrm{O}}^{\mathrm{A'}} \tfrac{1}{2}x\theta . x\theta\,\delta A + \int_{\mathrm{O}}^{\mathrm{B'}} \tfrac{1}{2}x\theta . x\theta\,\delta A$$

$$\therefore \qquad V\,(\bar{z}' - \bar{z}) = \tfrac{1}{2}\theta^2 \int_{\mathrm{A'}}^{\mathrm{B'}} x^2 dA. \qquad 16.9.1$$

Thus, as previously asserted, the vertical displacement of the centre of buoyancy is of order θ^2, and, to first order in θ may be neglected.

Now consider the moments of volume about the y-axis. We have

$$V\,\bar{x}' = V\,\bar{x} - \int_{\mathrm{O}}^{\mathrm{A'}} x\theta\,dA\,x + \int_{\mathrm{O}}^{\mathrm{A'}} x\theta\,dA\,x$$

$$\therefore \qquad V\,(\bar{x}' - \bar{x}) = \theta \int_{\mathrm{A'}}^{\mathrm{B'}} x^2 dA. \qquad 16.9.2$$

But the integral on the right hand side of equation 16.9.2 is Ak^2, where A is the area of the water-line section, and k is its radius of gyration.

Thus $$\mathrm{HH'} = \frac{Ak^2\theta}{V}. \qquad 16.9.3$$

Now $\mathrm{HH'} = \mathrm{HM}\sin\theta$, where M is the metacentre, or, to first order in θ, $\mathrm{HH'} = \mathrm{HM} \times \theta$.

$$\therefore \qquad \mathrm{HM} = \frac{Ak^2}{V}. \qquad 16.9.4$$

Therefore the *condition for stability of equilibrium* is that

$$\frac{Ak^2}{V} > \mathrm{HC}. \qquad 16.9.5$$

Here, A and k^2 refer to the water-line section, V is the volume submerged, and HC is the distance between centre of buoyancy and centre of mass.

Example. Suppose that the body is a cube of side $2a$ and of relative density s. The water-line section is a square, and $A = 4a^2$ and $k^2 = a^2/3$. The volume submerged is $8a^3s$. The distance between the centres of mass and buoyancy is $a(1 - s)$, and so the condition for stability is

$$\frac{a}{6s} > a(1-s). \qquad 16.9.6$$

The equilibrium is unstable if

$$6s^2 - 6s + 1 < 0. \qquad 16.9.7$$

That is, the equilibrium is unstable if s is between 0.2113 and 0.7887. The cube will float vertically only if the density is less than 0.2113 or if it is greater than 0.7887.

Problem 9

Here in British Columbia there is a large logging industry, and many logs float horizontally in the water. They gradually become waterlogged, and, when the density of a log is nearly as dense as the water, the vertical position become stable and the log tips to the vertical position, nearly all of it submerged, with only an inch or so above the surface. It then becomes a danger to boats. If the length of the log is $2l$ and its radius is a, what is the least relative density for which the vertical position is stable?

Solutions to Problems 8 and 9

Solution to problem 8

Let us establish some notation.
V = total volume of each body
fV = volume of liquid displaced by the first body (i.e. volume below the waterline in the first drawing)
$(1-f)V$ = volume of liquid displaced by the second body (i.e. volume below the waterline in the second drawing)
ρ_0 = density of liquid
ρ_1 = density of first body = $s_1\rho_0$
ρ_2 = density of first body = $s_2\rho_0$
g = gravitational acceleration

Now:

Weight of first body = weight of liquid displaced: $V\rho_1 g = fV\rho_0 g$ I.e. $s_1 = f$

Weight of second body = weight of liquid displaced: $V\rho_2 g = (1-f)V\rho_0 g$

I.e. $s_2 = 1 - f$

Hence $\underline{\underline{s_2 = 1 - f.}}$

Solution to problem 9

The *condition for stability of equilibrium* is that

$$\frac{Ak^2}{V} > \text{HC}.$$

Here, A and k^2 refer to the water-line section, V is the volume submerged, and HC is the distance between centre of mass and centre of buoyancy.

In the present case we have a log of radius a and length $2l$. In this case

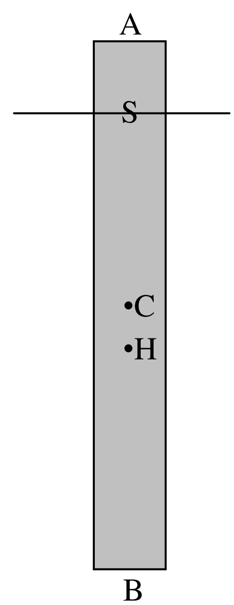
$$A = \pi a^2, \quad k^2 = \tfrac{1}{2}a^2, \quad V = 2\pi a^2 l.$$

$$\frac{Ak^2}{V} = \frac{a^2}{4l}$$

Density of log = ρ

Density of water = ρ_0

Relative density $s = \rho/\rho_0$



Some distances:

AB = $2l$

AC = l

SB = $2ls$

AS = $2l(1-s)$

SH = ls

SC = AC − AS = $2ls - l$

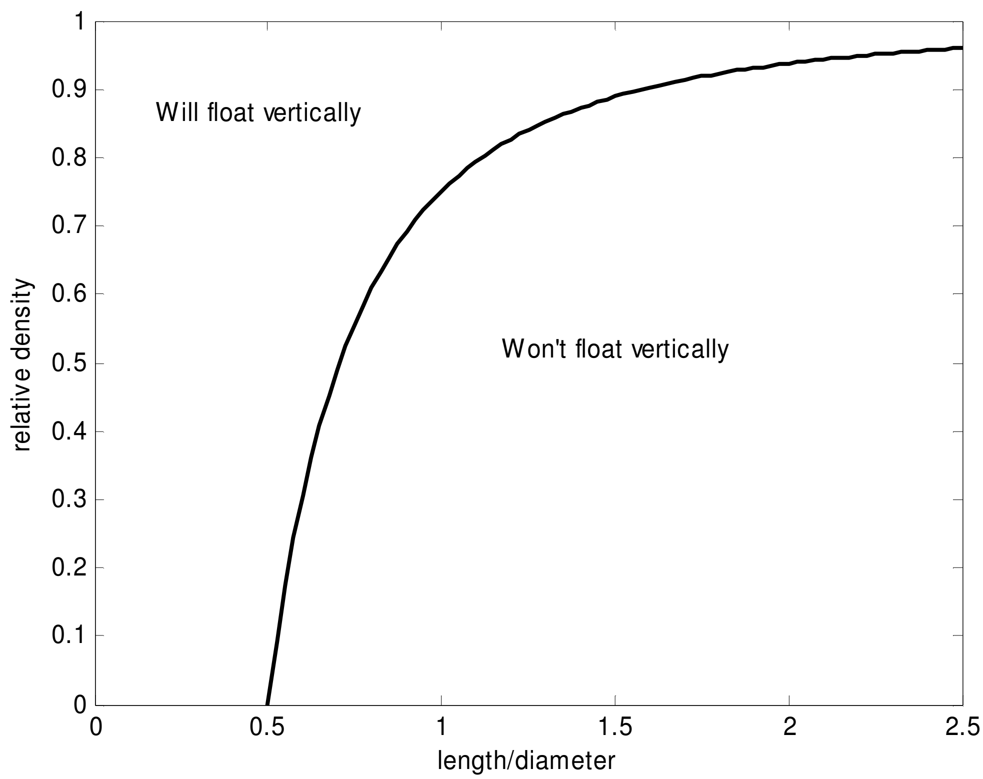
HC = SH − SC = $l(1-s)$

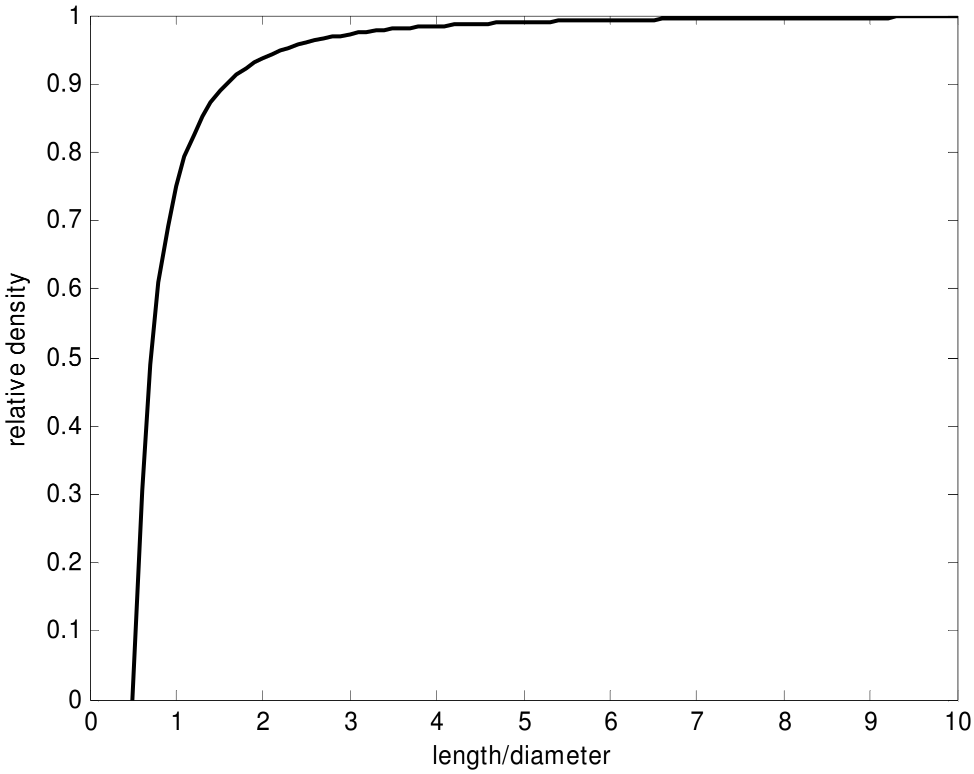
The condition for stability is that $\dfrac{a^2}{4l} > l(1-s)$

That is: $s > 1 - \dfrac{1}{4}\left(\dfrac{\text{diameter}}{\text{length}}\right)^2.$

length/diameter	=	0.5	0.71	1	2	10	40
relative density	>	0	0.50	0.75	0.9375	0.9975	0.9998

A flat log, whose length is less than half its diameter, floats with its axis vertical, whatever its density (provided, of course, that it is less than that of water, when it will sink). If its length is equal to its diameter, it will float vertically provided that its density is at least 0.75 that of water. A very long log floats horizontally until it is almost completely saturated with water, and then it will tip over to a vertical position, almost completely submerged, when it is not readily visible and it is then a danger to boats. The condition for vertically-floating stable equilibrium is illustrated in the two graphs below.




1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
relative density
0
1
2
3
4
5
6
7
8
9
10
length/diameter