CHAPTER 13
LAGRANGIAN MECHANICS

13.1 *Introduction*

The usual way of using newtonian mechanics to solve a problem in dynamics is first of all to draw a large, clear diagram of the system, using a ruler and a compass. Then mark in the forces on the various parts of the system with red arrows and the accelerations of the various parts with green arrows. Then apply the equation $F = ma$ in two different directions if it is a two-dimensional problem or in three directions if it is a three-dimensional problem, or $\tau = I\ddot{\theta}$ if torques are involved. More correctly, if a mass or a moment of inertia is not constant, the equations are $F = \dot{p}$ and $\tau = \dot{L}$. In any case, we arrive at one or more *equations of motion*, which are differential equations which we integrate with respect to space or time to find the desired solution. Most of us will have done many, many problems of that sort.

Sometimes it is not all that easy to find the equations of motion as described above. There is an alternative approach known as lagrangian mechanics which enables us to find the equations of motion when the newtonian method is proving difficult. In lagrangian mechanics we start, as usual, by drawing a large, clear diagram of the system, using a ruler and a compass. But, rather than drawing the forces and accelerations with red and green arrows, we draw the *velocity* vectors (including angular velocities) with blue arrows, and, from these we write down the *kinetic energy* of the system. If the forces are *conservative* forces (gravity, springs and stretched strings), we write down also the *potential energy*. That done, the next step is to write down the *lagrangian equations of motion* for each coordinate. These equations involve the kinetic and potential energies, and are a little bit more involved than $F = ma$, though they do arrive at the same results.

I shall derive the lagrangian equations of motion, and while I am doing so, you will think that the going is very heavy, and you will be discouraged. At the end of the derivation you will see that the lagrangian equations of motion are indeed rather more involved than $F = ma$, and you will begin to despair – but do not do so! In a very short time after that you will be able to solve difficult problems in mechanics that you would not be able to start using the familiar newtonian methods, and the speed at which you do so will be limited solely by the speed at which you can write. Indeed, you scarcely have to stop and think. You know straight away what you have to do. Draw the diagram. Mark the velocity vectors. Write down expressions for the kinetic and potential energies, and apply the lagrangian equations. It is automatic, fast, and enjoyable.

Incidentally, when Lagrange first published his great work *La méchanique analytique* (the modern French spelling would be *mécanique*), he pointed out with some pride in his introduction that there were no drawings or diagrams in the book – because all of mechanics could be done *analytically* – i.e. with algebra and calculus. Not all of us, however, are as gifted as Lagrange, and we cannot omit the first and very important step

of drawing a large and clear diagram with ruler and compass and marking all the velocity vectors.

13.2 *Generalized Coordinates and Generalized Forces*

In two-dimensions the positions of a point can be specified either by its *rectangular coordinates* (x, y) or by its *polar coordinates*. There are other possibilities such as confocal conical coordinates that might be less familiar. In three dimensions there are the options of *rectangular coordinates* (x, y, z), or *cylindrical coordinates* (ρ, ϕ, z) or *spherical coordinates* (r, θ , ϕ) – or again there may be others that may be of use for specialized purposes (inclined coordinates in crystallography, for example, come to mind). The state of a molecule might be described by a number of parameters, such as the bond lengths and the angles between the bonds, and these may be varying periodically with time as the molecule vibrates and twists, and these bonds lengths and bond angles constitute a set of *coordinates* which describe the molecule. We are not going to think about any particular sort of coordinate system or set of coordinates. Rather, we are going to think about *generalized coordinates*, which may be lengths or angles or various combinations of them. We shall call these coordinates (q_1 , q_2 , q_3 ,...). If we are thinking of a single particle in three-dimensional space, there will be three of them, which could be rectangular, or cylindrical, or spherical. If there were N particles, we would need $3N$ coordinates to describe the system – unless there were some constraints on the system.

With each generalized coordinate q_j is associated a *generalized force* P_j, which is defined as follows. If the work required to increase the coordinate q_j by δq_j is $P_j\ \delta q_j$, then P_j is the generalized force associated with the coordinate q_j.

It will be noted that a generalized force need not always be dimensionally equivalent to a force. For example, if a generalized coordinate is an angle, the corresponding generalized force will be a torque.

One of the things that we shall want to do is to identify the generalized force associated with a given generalized coordinate.

13.3 *Holonomic constraints*

The complete description of a system of N unconstrained particles requires $3N$ coordinates. You can think of the state of the system at any time as being represented by a single point in $3N$-dimensional space. If the system consists of molecules in a gas, or a cluster of stars, or a swarm of bees, the coordinates will be continually changing, and the point that describes the system will be moving, perhaps completely unconstrained, in its $3N$-dimensional space.

However, in many systems, the particles may not be free to wander anywhere at will; they may be subject to various *constraints*. A constraint that can be described by an equation relating the coordinates (and perhaps also the time) is called a *holonomic constraint*, and the equation that describes the constraint is a *holonomic equation.* If a system of N particles is subject to k holonomic constraints, the point in $3N$-dimensional space that describes the system at any time is not free to move anywhere in $3N$-dimensional space, but it is constrained to move over a surface of dimension $3N - k$. In effect only $3N - k$ coordinates are needed to describe the system, given that the coordinates are connected by k holonomic equations.

Incidentally, I looked up the word "holonomic" in *The Oxford English Dictionary* and it said that the word was from the Greek ὅλος, meaning "whole" or "entire" and νόμ-ος, meaning "law". It also said "applied to a constrained system in which the equations defining the constraints are integrable or already free of differentials, so that each equation effectively reduces the number of coordinates by one; also applied to the constraints themselves."

As an example, consider a bar of wet soap slithering around in a hemispherical basin of radius a. You can describe its position in the basin by means of the usual two spherical angles (θ , ϕ); the motion is otherwise constrained by its remaining in contact with the basin; that is to say it is subject to the holonomic constraint $r = a$. Thus instead of needing three coordinates to describe the position of a totally unconstrained particle, we need only two coordinates.

Or again, consider the double pendulum shown in figure XIII.1, and suppose that the pendulum is constrained to swing only in the plane of the paper – or of the screen of your computer monitor.

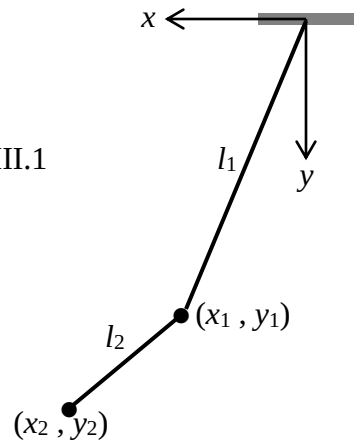


FIGURE XIII.1

Two unconstrained particles would require six coordinates to specify their positions but this system is subject to four holonomic constraints. The holonomic equations $z_1 = 0$

and $z_2 = 0$ constrain the particles to be moving in a plane, and, if the strings are kept taut, we have the additional holonomic constraints $x_1^2 + y_1^2 = l_1^2$ and $(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$. Thus only two coordinates are needed to describe the system, and they could conveniently be the angles that the two strings make with the vertical.

13.4 *The Lagrangian Equations of Motion*

This section might be tough – but don't be put off by it. I promise that, after we have got over this section, things will be easy. But in this section I don't like all these summations and subscripts any more than you do.

Suppose that we have a system of N particles, and that the force on the ith particle (i = 1 to N) is $\mathbf{F}_i$. If the ith particle undergoes a displacement $\delta\mathbf{r}_i$, the total work done on the system is $\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i$. The position vector $\mathbf{r}$ of a particle can be written as a function of its generalized coordinates; and a change in $\mathbf{r}$ can be expressed in terms of the changes in the generalized coordinates. Thus the total work done on the system is

$$\sum_i \mathbf{F}_i \cdot \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j , \qquad 13.4.1$$

which can be written

$$\sum_j \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j . \qquad 13.4.2$$

But by definition of the generalized force, the work done on the system is also

$$\sum_j P_j \cdot \delta q_j . \qquad 13.4.3$$

Thus the generalized force P_j associated with generalized coordinate q_j is given by

$$P_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} . \qquad 13.4.4$$

Now $\mathbf{F}_i = m_i \ddot{\mathbf{r}}_i$, so that

$$P_j = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} . \qquad 13.4.5$$

Also $$\frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\right) = \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} + \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right). \qquad 13.4.6$$

Substitute for $\ddot{\mathbf{r}} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$ from equation 13.4.6 into equation13.4.5 to obtain

$$P_j = \sum_i m_i \left[\frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\right) - \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right)\right]. \qquad 13.4.7$$

Now $\frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}$ and $\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}$.

Therefore $$P_j = \sum_i m_i \left[\frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}\right) - \dot{\mathbf{r}}_i \cdot \left(\frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}\right)\right]. \qquad 13.4.8$$

You may not be immediately comfortable with the assertions

$\frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}$ and $\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}$ so I'll interrupt the flow briefly here to try to justify these assertions and to understand what they mean. As an example, consider the relation between the coordinate *x* and the spherical coordinates *r*, θ , ϕ:

$$x = r\sin\theta\cos\phi. \qquad (A1)$$

In this example, *x* would correspond to one of the components of $\mathbf{r}_i$, and *r*, θ , ϕ are the q_1, q_2, q_3.

From equation (A1), we easily derive

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \qquad \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \qquad \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi \qquad (A2)$$

On differentiating equation (A1) with respect to time, we obtain

$$\dot{x} = \dot{r}\sin\theta\cos\phi + r\cos\theta\dot{\theta}\cos\phi - r\sin\theta\sin\phi\dot{\phi} \qquad (A3)$$

And from this we see that

$$\frac{\partial \dot{x}}{\partial \dot{r}} = \sin\theta\cos\phi \qquad \frac{\partial \dot{x}}{\partial \dot{\theta}} = r\cos\theta\cos\phi \qquad \frac{\partial \dot{x}}{\partial \dot{\phi}} = -r\sin\theta\sin\phi \qquad \text{(A4)}$$

Thus the first assertion is justified in this example, and I think you'll see that it will always be true no matter what the functional dependence of $\mathbf{r}_i$ on the q_j.

For the second assertion, consider

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \quad \text{and hence} \quad \frac{d}{dt}\frac{\partial x}{\partial r} = \cos\theta\dot{\theta}\cos\phi - \sin\theta\sin\phi\dot{\phi}. \qquad \text{(A5)}$$

From equation (A3) we find that

$$\frac{\partial \dot{x}}{\partial r} = \cos\theta\dot{\theta}\cos\phi - \sin\theta\sin\phi\dot{\phi}, \qquad \text{(A6)}$$

and the second assertion is justified. Again, I think you'll see that it will always be true no matter what the functional dependence of $\mathbf{r}_i$ on the q_j.

We continue.

The kinetic energy T is

$$T = \sum_i \tfrac{1}{2} m_i \dot{r}_i^2 = \sum_i \tfrac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i . \qquad 13.4.9$$

Therefore

$$\frac{\partial T}{\partial q_j} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \qquad 13.4.10$$

and

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} . \qquad 13.4.11$$

On substituting these in equation 13.4.8 we obtain

$$\underline{\underline{P_j = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}}} . \qquad 13.4.12$$

This is one form of Lagrange's equation of motion, and it often helps us to answer the question posed in the last sentence of section 13.2 – namely to determine the generalized force associated with a given generalized coordinate.

If the various forces in a particular problem are *conservative* (gravity, springs and stretched strings, including valence bonds in a molecule) then the generalized force can be obtained by the negative of the gradient of a potential energy function – i.e. $P_j = -\frac{\partial V}{\partial q_j}$. In that case, Lagrange's equation takes the form

$$\underline{\underline{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}}}. \qquad 13.4.13$$

In my experience this is the most useful and most often encountered version of Lagrange's equation.

The quantity $L = T - V$ is known as the *lagrangian* for the system, and Lagrange's equation can then be written

$$\underline{\underline{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0}}. \qquad 13.4.14$$

This form of the equation is seen more often in theoretical discussions than in the practical solution of problems. It does enable us to see one important result. If, for one of the generalized coordinates, $\frac{\partial L}{\partial q_j} = 0$ (this could happen if neither *T* nor *V* depends on q_j – but of course it could also happen if $\partial T / \partial q_j$ and $\partial V / \partial q_j$ were nonzero but equal and opposite in sign), then that generalized coordinate is called an *ignorable coordinate* – presumably because one can ignore it in setting up the lagrangian. However, it doesn't really mean that it should be ignored altogether, because it *immediately reveals a constant* of the motion. In particular, if $\frac{\partial L}{\partial q_j} = 0$, then $\frac{\partial L}{\partial \dot{q}_j}$ *is constant*. It will be seen that if q_j has the dimensions of length, $\frac{\partial L}{\partial \dot{q}_j}$ has the dimensions of linear momentum. And if q_i is an angle, $\frac{\partial L}{\partial \dot{q}_j}$ has the dimensions of angular momentum. The derivative $\frac{\partial L}{\partial \dot{q}_j}$ is usually given the symbol p_j and is called *the generalized momentum conjugate to the*

generalized coordinate q_j. If q_j is an "ignorable coordinate", then p_j is a constant of the motion.

In each of equations 13.4.12, 13 and 14 one of the qs has a dot over it. You can see which one it is by thinking about the *dimensions* of the various terms. Dot has dimension T^{-1}.

So, we have now derived Lagrange's equation of motion. It was a hard struggle, and in the end we obtained three versions of an equation which at present look quite useless. But from this point, things become easier and we rapidly see how to use the equations and find that they are indeed very useful.

13.5 *Acceleration Components*

In section 3.4 of chapter 3 of the Celestial Mechanics "book", I derived the radial and transverse components of velocity and acceleration in two-dimensional coordinates. The radial and transverse velocity components are fairly obvious and scarcely need derivation; they are just $\dot{\rho}$ and $\rho\dot{\phi}$. For the acceleration components I reproduce here an extract from that chapter:

"The radial and transverse components of acceleration are therefore $(\ddot{\rho} - \rho\dot{\phi}^2)$ and $(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})$ respectively."

I also derived the radial, meridional and azimuthal components of velocity and acceleration in three-dimensional spherical coordinates. Again the velocity components are rather obvious; they are $\dot{r}$, $r\dot{\theta}$ and $r\sin\theta\dot{\phi}$, while for the acceleration components I reproduce here the relevant extract from that chapter.

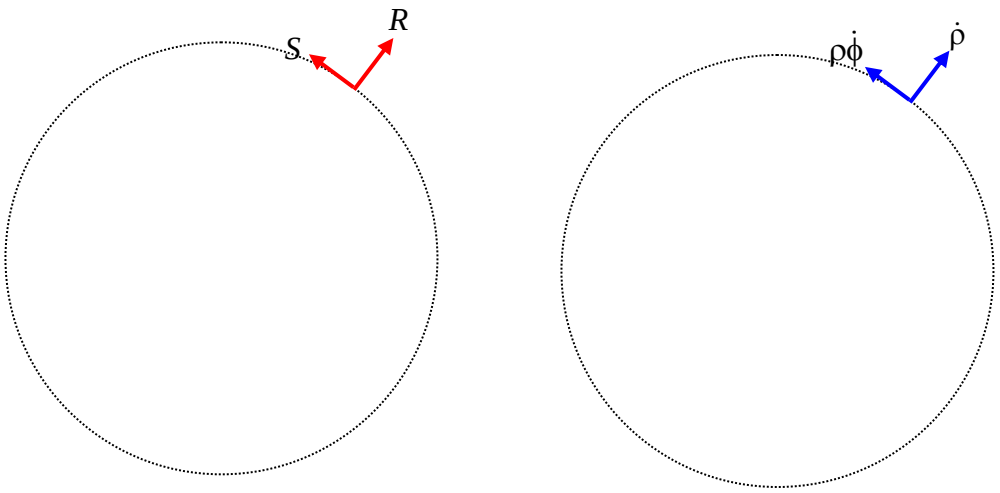
"On gathering together the coefficients of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$, we find that the components of acceleration are:

Radial: $\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2$

Meridional: $r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2$

Azimuthal: $2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\ddot{\phi}$. "

You might like to look back at these derivations now. However, I am now going to derive them by a different method, using Lagrange's equation of motion. You can decide for yourself which you prefer.



We'll start in two dimensions. Let R and S be the radial and transverse components of a force acting on a particle. ("Radial" means in the direction of increasing ρ; "transverse" means in the direction of increasing ϕ.) If the radial coordinate were to increase by $\delta\rho$, the work done by the force would be just $R\,\delta\rho$. Thus the generalized force associated with the coordinate ρ is just $P_\rho = R$. If the azimuthal angle were to increase by $\delta\phi$, the work done by the force would be $S\rho\,\delta\phi$. Thus the generalized force associated with the coordinate ϕ is $P_\phi = S\rho$. Now we don't have to think about how to start; in Lagrangian mechanics, the first line is always "T = ...", and I hope you'll agree that

$$T = \tfrac{1}{2}m(\dot\rho^2 + \rho^2\dot\phi^2). \qquad 13.5.1$$

If you now apply equation 13.4.12 in turn to the coordinates ρ and ϕ, you obtain

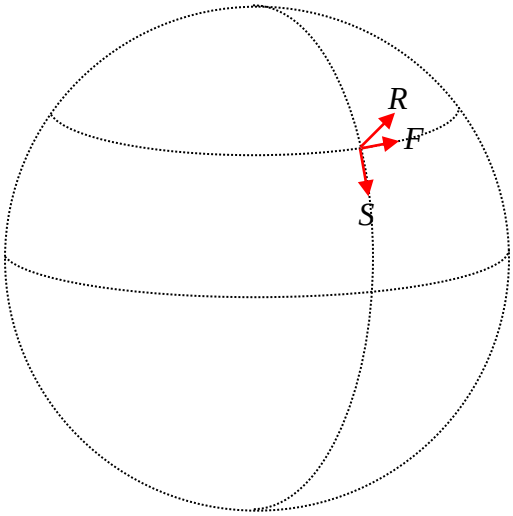
$$P_\rho = m(\ddot\rho - \rho\dot\phi^2) \quad \text{and} \quad P_\phi = m\rho(\rho\ddot\phi + 2\dot\rho\dot\phi), \qquad 13.5.2a,b$$

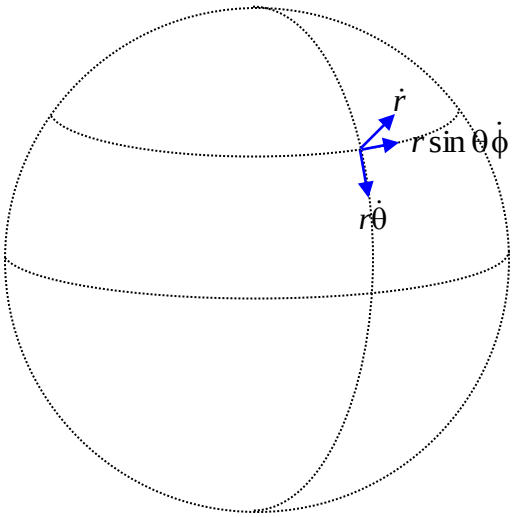
and so

$$R = m(\ddot\rho - \rho\dot\phi^2) \quad \text{and} \quad S = m(\rho\ddot\phi + 2\dot\rho\dot\phi). \qquad 13.5.3a,b$$

Therefore the radial and transverse components of the acceleration are $(\ddot\rho - \rho\dot\phi^2)$ and $(\rho\ddot\phi + 2\dot\rho\dot\phi)$ respectively.

We can do exactly the same thing to find the acceleration components in three-dimensional spherical coordinates. Let R , S and F be the radial, meridional and azimuthal (i.e. in direction of increasing r, θ and ϕ) components of a force on a particle.





If r increases by δr, the work on the particle done is $R\,\delta r$.
If θ increases by $\delta\theta$, the work done on the particle is $Sr\,\delta\theta$.
If ϕ increases by $\delta\phi$, the work done on the particle is $Fr\sin\theta\,\delta\phi$.

Therefore $P_r = R$, $P_\theta = Sr$ and $P_\phi = Fr\sin\theta$.

Start: $$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2). \qquad 13.5.4$$

If you now apply equation 13.4.12 in turn to the coordinates r, θ and ϕ, you obtain

$$P_r = m\left(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2\right) \qquad 13.5.5$$

$$P_\theta = m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\dot{\phi}^2) \qquad 13.5.6$$

and $$P_\phi = m(r^2\sin^2\theta\ddot{\phi} + 2r^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta + 2r\dot{r}\dot{\phi}\sin^2\theta). \qquad 13.5.7$$

Therefore

$$R = m\left(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2\right) \qquad 13.5.8$$

$$S = m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2) \qquad 13.5.9$$

and $$F = m(r\sin\theta\ddot{\phi} + 2r\dot{\theta}\dot{\phi}\cos\theta + 2\dot{r}\dot{\phi}\sin\theta). \qquad 13.5.10$$

Thus the acceleration components are

Radial: $\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2$

Meridional: $r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2$

Azimuthal: $2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\,\ddot{\phi}$.

Be sure to check the dimensions. Since dot has dimension T^{-1}, and these expressions must have the dimensions of acceleration, there must be an r and two dots in each term.

13.6 *Slithering Soap in Conical Basin*

We imagine a slippery (no friction) bar of soap slithering around in a conical basin. An isolated bar of soap in intergalactic space would require three coordinates to specify its position at any time, but, if it is subject to the holonomic constraint that it is to be in contact at all times with a conical basin, its position at any time can be specified with just two coordinates. I shall, first of all, analyse the problem with a newtonian approach, and then, for comparison, I shall analyse it using lagrangian methods. Either way, we start with a large diagram. In the newtonian approach we mark in the forces in red and the accelerations in green. See figure XIII.2. The semi vertical angle of the cone is α.

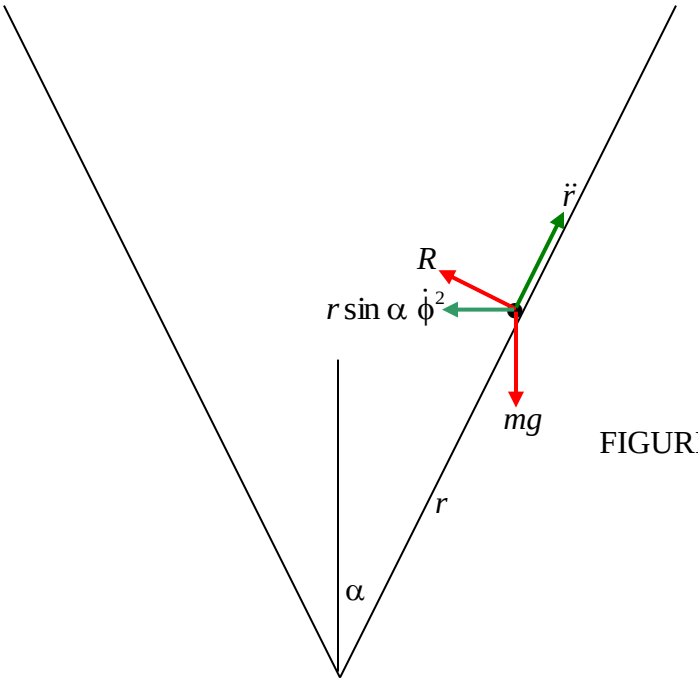


FIGURE XIII.2

The two coordinates that we need are r, the distance from the vertex, and the azimuthal angle ϕ, which I'll ask you to imagine, measured around the vertical axis from some arbitrary origin. The two forces are the weight mg and the normal reaction R of the basin on the soap. The accelerations are $\ddot{r}$ and the centripetal acceleration as the soap moves at angular speed $\dot{\phi}$ in a circle of radius $r\sin\alpha$ is $r\sin\alpha\,\dot{\phi}^2$.

We can write the newtonian equation of motion in various directions:

Horizontal: $$R\cos\alpha = m(r\sin\alpha\dot{\phi}^2 - \ddot{r}\sin\alpha)$$

i.e. $$R = m\tan\alpha(r\dot{\phi}^2 - \ddot{r}). \qquad 13.6.1$$

Vertical: $$R\sin\alpha - mg = m\ddot{r}\cos\alpha. \qquad 13.6.2$$

Perpendicular to surface: $$R - mg\sin\alpha = mr\sin\alpha\cos\alpha\,\dot{\phi}^2. \qquad 13.6.3$$

Parallel to surface: $$g\cos\alpha = r\sin^2\alpha\,\dot{\phi}^2 - \ddot{r}. \qquad 13.6.4$$

Only two of these are independent, and we can choose to use whichever two we want to at our convenience. There are, however, three quantities that we may wish to determine, namely the two coordinates r and ϕ, and the normal reaction R. Thus we need another equation. We note that, since there are no azimuthal forces, the angular momentum per unit mass, which is $r^2\sin^2\alpha\,\dot{\phi}$, is conserved, and therefore $r^2\dot{\phi}$ is constant and equal to its initial value, which I'll call $l^2\Omega$. That is, we start off at a distance l from the vertex with an initial angular speed Ω. Thus we have as our third independent equation

$$r^2\dot{\phi} = l^2\Omega. \qquad 13.6.5$$

This last equation shows that $\dot{\phi} \to \infty$ as $r \to 0$.

One possible type of motion is circular motion at constant height (put $\ddot{r} = 0$). From equations 13.6.1 and 2 it is easily found that the condition for this is that

$$r\dot{\phi}^2 = \frac{g}{\sin\alpha\tan\alpha}. \qquad 13.6.6$$

In other words, if the particle is projected initially horizontally ($\dot{r} = 0$) at $r = l$ and $\dot{\phi} = \Omega$, it will describe a horizontal circle (for ever) if

$$\Omega = \left(\frac{g}{l\sin\alpha\tan\alpha}\right)^{1/2} = \Omega_C, \text{ say.} \qquad 13.6.7$$

If the initial speed is less than this, the particle will describe an elliptical orbit with a minimum $r < l$; if the initial speed is greater than this, the particle will describe an elliptical orbit with a maximum $r > l$.

Now let's do the same problem in a lagrangian formulation. This time we draw the same diagram, but we mark in the velocity components in blue. See figure XIII.3. We are dealing with conservative forces, so we are going to use equation 13.4.13, the most useful form of Lagrange's equation.

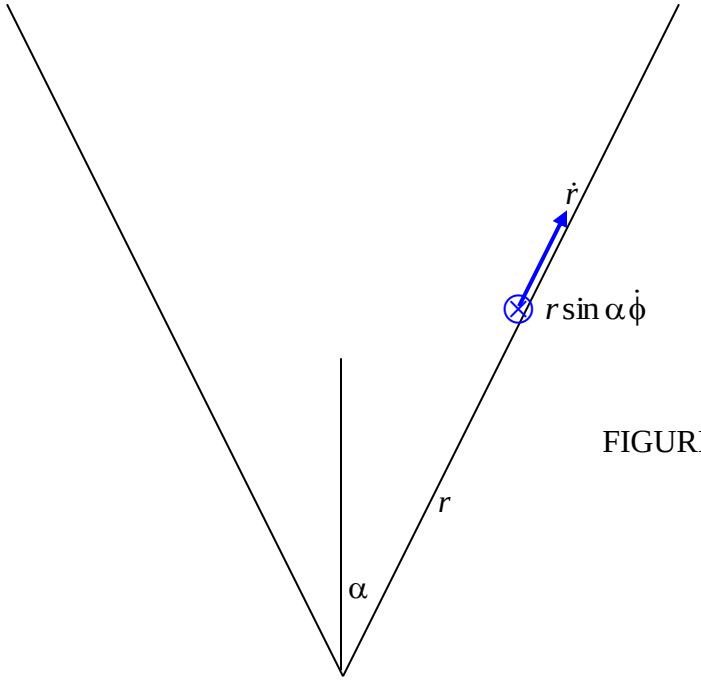


FIGURE XIII.3

We need not spend time wondering what to do next. The first and second things we always have to do are to find the kinetic energy T and the potential energy V, in order that we can use equation 13.4.13.

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2) \qquad 13.6.8$$

and

$$V = mgr\cos\alpha + \text{constant}. \qquad 13.6.9$$

Now go to equation 13.4.13, with $q_i = r$, and work out all the derivatives, and you should get, when you apply the lagrangian equation to the coordinate r:

$$\ddot{r} - r\sin^2\alpha\,\dot{\phi}^2 = -g\cos\alpha. \qquad 13.6.10$$

Now do the same thing with the coordinate ϕ. You see immediately that $\frac{\partial T}{\partial \phi}$ and $\frac{\partial V}{\partial \phi}$ are both zero. Therefore $\frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}}$ is zero and therefore $\frac{\partial T}{\partial \dot{\phi}}$ is constant. That is, $mr^2 \sin^2 \alpha \dot{\phi}$ is constant and so $r^2\dot{\phi}$ is constant and equal to its initial value $l^2\Omega$. Thus the second lagrangian equation is

$$r^2\dot{\phi} = l^2\Omega. \qquad 13.6.11$$

Since the lagrangian is independent of ϕ, ϕ is called, in this connection, an "ignorable coordinate" – and the momentum associated with it, namely $mr^2\dot{\phi}$, is constant.

Now it is true that we arrived at both of these equations also by the newtonian method, and you may not feel we have gained much. But this is a simple, introductory example, and we shall soon appreciate the power of the lagrangian method,

Having got these two equations, whether by newtonian or lagrangian methods, let's explore them further. For example, let's eliminate $\dot{\phi}$ between them and hence get a single equation in *r*:

$$\ddot{r} - \frac{l^4\Omega^2 \sin^2 \alpha}{r^3} = -g\cos\alpha. \qquad 13.6.12$$

We know enough by now (see Chapter 6) to write $\ddot{r}$ as $v\frac{dv}{dr}$, where $v = \dot{r}$, and if we let the constants $l^4\Omega^2 \sin^2 \alpha$ and $g \cos \alpha$ equal *A* and *B* respectively, equation 13.6.12 becomes

$$v\frac{dv}{dr} = \frac{A}{r^3} - B. \qquad 13.6.13$$

(It may just be useful to note that the dimensions of *A* and *B* are L^4T^{-2} and LT^{-2} respectively. This will enable us to keep track of dimensional analysis as we go.)

If we start the soap moving horizontally ($v = 0$) when $r = l$, this integrates, with these initial conditions, to

$$v^2 = A\left(\frac{1}{l^2} - \frac{1}{r^2}\right) + 2B(l - r). \qquad 13.6.14$$

Again, so that we can see what we are doing, let $\frac{A}{l^2} + 2Bl = C$ (note that $[C] = \mathrm{L}^2\mathrm{T}^{-2}$), and equation13.6.14 becomes

$$v^2 = C - \frac{A}{r^2} - 2Br. \qquad 13.6.15$$

This gives $v\ (=\dot{r})$ as a function of r. The particle reaches is maximum or minimum height when $v = 0$; that is where

$$2Br^3 - Cr^2 + A = 0. \qquad 13.6.16$$

One solution of this is obviously $r = l$. Of the other two solutions, one is positive (which we want) and the other is negative (which we don't want).

If we go back to the original meanings of A, B and C, and write $x = r/l$, equation (16) becomes, after a little tidying up

$$x^3 - \left(\frac{l\Omega^2 \sin\alpha \tan\alpha}{2g} + 1\right)x^2 + \frac{l\Omega^2 \sin\alpha \tan\alpha}{2g} = 0. \qquad 13.6.17$$

Recall from equation 13.6.7 that $\Omega_c = \left(\frac{g}{l \sin\alpha \tan\alpha}\right)^{1/2}$, and the equation becomes

$$x^3 - \left(\frac{\Omega^2}{2\Omega_c^2} + 1\right)x^2 + \frac{\Omega^2}{2\Omega_c^2} = 0, \qquad 13.6.18$$

or, with $a = \frac{\Omega^2}{2\Omega_c^2}$,

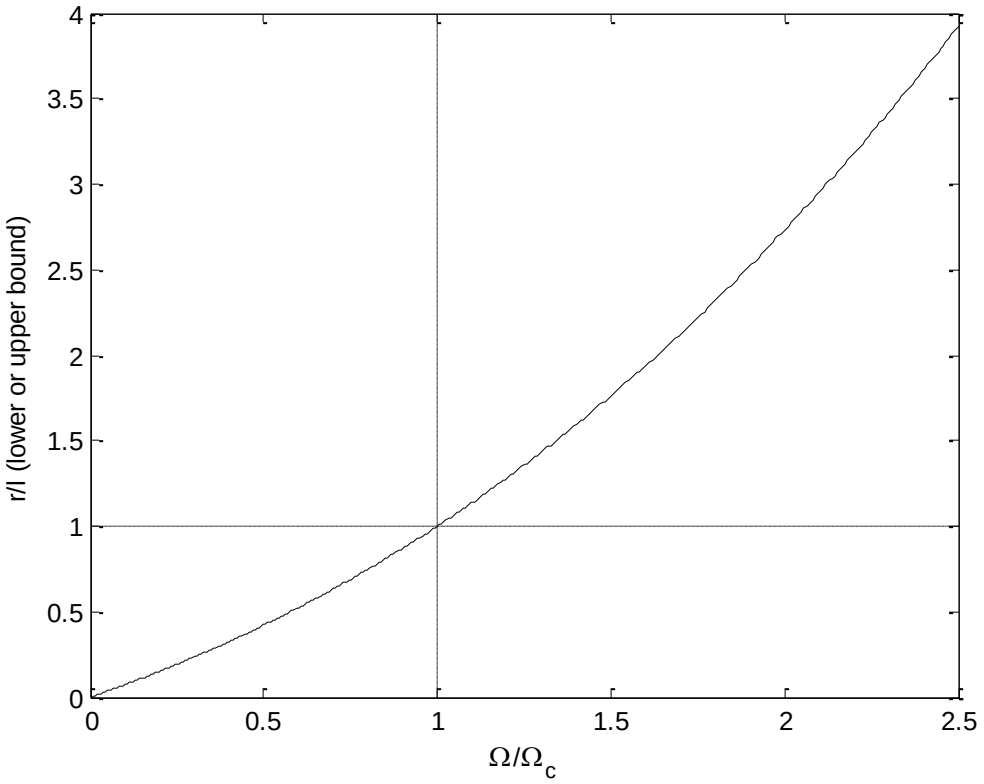
$$x^3 - (a + 1)x^2 + a = 0. \qquad 13.6.19$$

This factorizes to

$$(x - 1)(x^2 - ax - a) = 0. \qquad 13.6.20$$

The solution we are interested in is

$$x = \tfrac{1}{2}\left(a + \sqrt{a(a + 4)}\right). \qquad 13.6.21$$



13.7 *Slithering Soap in Hemispherical Basin*

Suppose that the basin is of radius *a* and the soap is subject to the holonomic constraint $r = a$ - i.e. that it remains in contact with the basin at all times. Note also that this is just the same constraint of a pendulum free to swing in three-dimensional space except that it is subject to the holonomic constraint that the string be taut at all times. Thus any conclusions that we reach about our soap will also be valid for a pendulum.

We'll start with the newtonian approach, and I'll draw in red the two forces on the soap, namely its weight and the normal reaction of the basin on the soap. Figure XIII.4

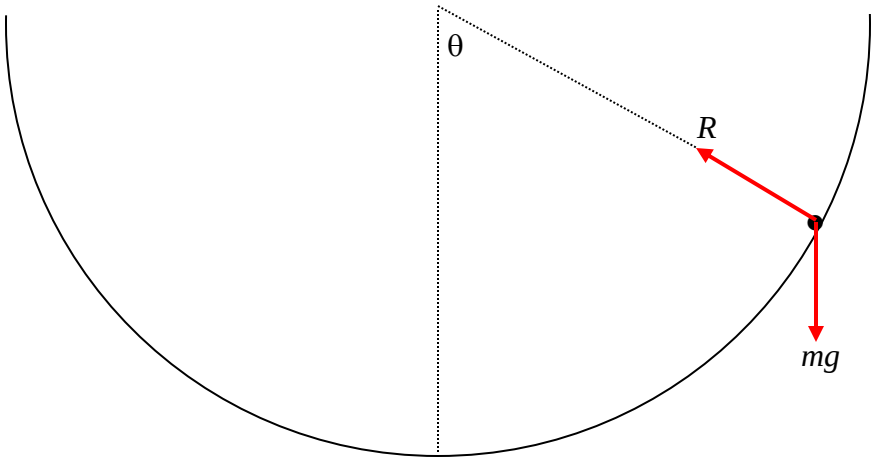


FIGURE XIII.4

We'll make use of the expressions for the radial, meridional and azimuthal accelerations from section 13.5 and we'll write down the equations of motion in these directions:

Radial: $$mg\cos\theta - R = m(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2), \qquad 13.7.1$$

Meridional: $$-mg\sin\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2), \qquad 13.7.2$$

Azimuthal: $$0 = m(2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\ddot{\phi}). \qquad 13.7.3$$

We also have the constraint that $r = a$ and hence that $\dot{r} = \ddot{r} = 0$, after which these equations become

$$mg\cos\theta - R = -ma(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2), \qquad 13.7.4$$

$$-g\sin\theta = a(\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2), \qquad 13.7.5$$

$$0 = \sin\theta\ddot{\phi} + 2\dot{\theta}\dot{\phi}\cos\theta. \qquad 13.7.6$$

These, then, are the newtonian equations of motion. If you still prefer the newtonian method to the lagrangian method, and you wish to integrate these and find expressions θ, ϕ and R separately, by all means go ahead and do so – but I'm now going to try the lagrangian approach.

Although Lagrange himself would not have drawn a diagram, we shall not omit that step – but instead of marking in the forces, we'll mark in the velocity components, and then we'll immediately write down expressions for the kinetic and potential energies. Indeed the first line of a lagrangian calculation is always "T = ...".

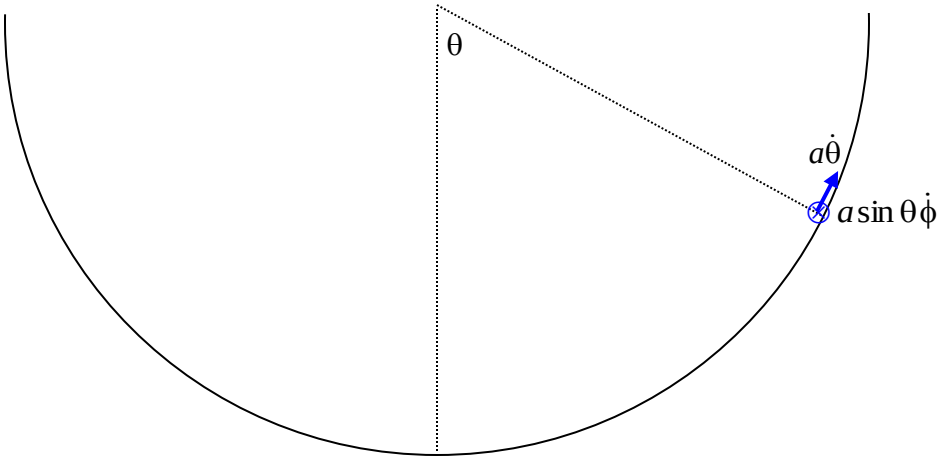


FIGURE XIII.5

$$T = \tfrac{1}{2}ma^2(\dot\theta^2 + \sin^2\theta\dot\phi^2). \qquad 13.7.7$$

$$V = -mga\cos\theta + \text{constant}. \qquad 13.7.8$$

Now apply equation 13.4.13 in turn to the coordinates θ and ϕ.

θ: $$a\ddot\theta - a\sin\theta\cos\theta\dot\phi^2 = -g\sin\theta. \qquad 13.7.9$$

ϕ: As for the conical basin, we see that $\frac{\partial T}{\partial\phi}$ and $\frac{\partial V}{\partial\phi}$ are both zero (ϕ is an "ignorable coordinate") and therefore $\frac{\partial T}{\partial\dot\phi}$ is constant and equal to its initial value. If the initial values of $\dot\phi$ and θ are Ω and α respectively, then

$$\sin^2\theta\dot\phi = \sin^2\alpha\Omega. \qquad 13.7.10$$

This is merely stating that angular momentum is conserved.

We can easily eliminate $\dot\phi$ from equations 13.4.9 and 10 to obtain

$$\ddot\theta = k\cot\theta\csc^2\theta - \frac{g\sin\theta}{a}, \qquad 13.7.11$$

where $$k = \sin^4\alpha\,\Omega^2. \qquad 13.7.12$$

Write $\ddot\theta$ as $\dot\theta\frac{d\dot\theta}{d\theta}$ in the usual way and integrate to obtain the first space integral:

$$\dot\theta^2 = \frac{2g}{a}(\cos\theta - \cos\alpha) - k(\csc^2\theta - \csc^2\alpha). \qquad 13.7.13$$

The upper and lower bounds for θ occur when $\dot\theta = 0$.

Example. Suppose that the initial value of θ is $\alpha = 45^o$ and that we start by pushing the soap horizontally ($\dot\theta = 0$) at an initial angular speed Ω = 3 rad s^{-1}, so that k = 2.25 rad^2 s^{-2}. Suppose that the radius of the basin is a = 1.96 m and that g = 9.8 m s^{-2}. You can then put $\dot\theta = 0$ in equation 13.7.13 and solve it for θ. One solution, of course, is θ = α. We could find the other solution by Newton-Raphson iteration, or by putting $\csc^2\theta = 1/(1-\cos^2\theta)$ and solving it as a cubic equation in cos θ. Alternatively, try this:

Let $\frac{ak}{2g} = n, \quad \cos\theta = x, \quad \cos\alpha = c,$

so that $\csc^2\theta = 1/(1-x^2)$ and $\csc^2\alpha = 1/(1-c^2)$.

The equation then becomes a quadratic equation in x, with solution

That is, $$x = \frac{-n + \sqrt{n^2 - 4(1-c^2)[c(n+c)-1]}}{2(1-c^2)}.$$

I'll leave you to re-write this in terms of what these quantities originally meant.

13.8 *More Examples*

i.

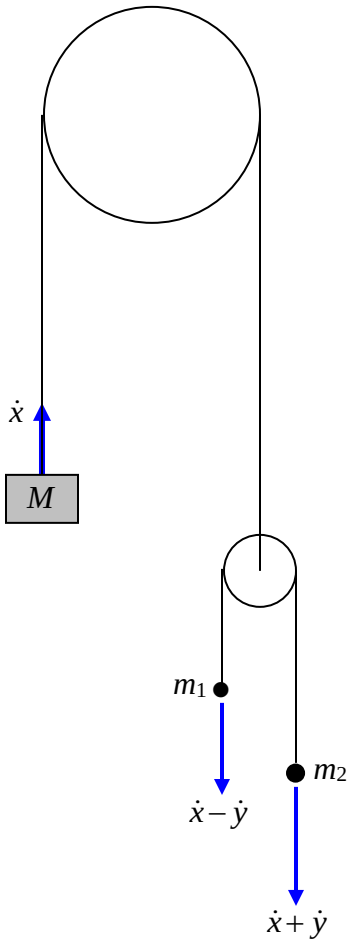


FIGURE XIII.6

The upper pulley is fixed in position. Both pulleys rotate freely without friction about their axles. Both pulleys are "light" in the sense that their rotational inertias are small and their rotation contributes negligibly to the kinetic energy of the system. The rims of the pulleys are rough, and the ropes do not slip on the pulleys. The gravitational acceleration is g. The mass M moves upwards at a rate $\dot{x}$ with respect to the upper, fixed, pulley, and the smaller pulley moves downwards at the same rate. The mass m_1 moves upwards at a rate $\dot{y}$ with respect to the small pulley, and consequently its speed in laboratory space is $\dot{x}-\dot{y}$. The speed of the mass m_2 is therefore $\dot{x}+\dot{y}$ in laboratory space. The object is to find $\ddot{x}$ and $\ddot{y}$ in terms of g.

The kinetic energy is $$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m_1(\dot{x}-\dot{y})^2 + \tfrac{1}{2}m_2(\dot{x}+\dot{y})^2. \qquad 13.8.1$$

The potential energy is $$V = g[Mx - m_1(x-y) - m_2(x+y)] + \text{constant}. \qquad 13.8.2$$

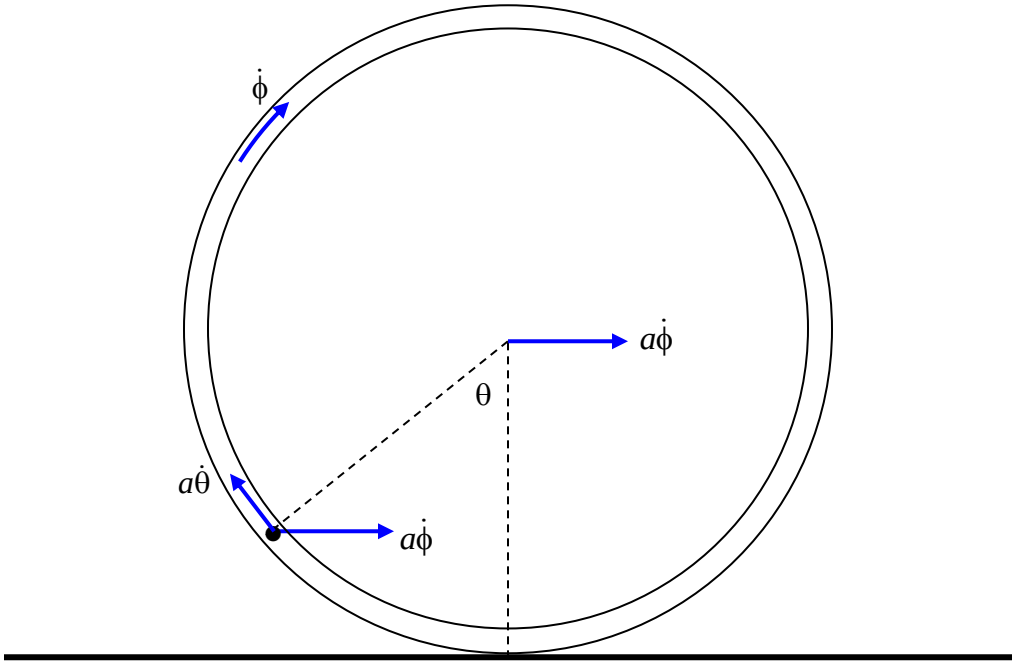
Apply Lagrange's equation (13.4.13) in turn to the coordinates x and y:

x: $$M\ddot{x} + m_1(\ddot{x}-\ddot{y}) + m_2(\ddot{x}+\ddot{y}) = -g(M-m_1-m_2). \qquad 13.8.3$$

y: $$-m_1(\ddot{x}-\ddot{y}) + m_2(\ddot{x}+\ddot{y}) = -g(m_1-m_2). \qquad 13.8.4$$

These two equations can be solved at one's leisure for $\ddot{x}$ and $\ddot{y}$.

ii. A torus of mass M and radius a rolls without slipping on a horizontal plane. A pearl of mass m slides smoothly around inside the torus. Describe the motion.



I have marked in the several velocity vectors. The torus is rolling at angular speed $\dot{\phi}$. Consequently the linear speed of the centre of mass of the hoop is $a\dot{\phi}$, and the pearl also shares this velocity. In addition, the pearl is sliding relative to the torus at an angular speed $\dot{\theta}$ and consequently has a component to its velocity of $a\dot{\theta}$ tangential to the torus. We are now ready to start.

The kinetic energy of the torus is the sum of its translational and rotational kinetic energies:

$$\tfrac{1}{2}M(a\dot{\phi})^2 + \tfrac{1}{2}(Ma^2)\dot{\phi}^2 = Ma^2\dot{\phi}^2.$$

The kinetic energy of the pearl is

$$\tfrac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2 - 2\dot{\theta}\dot{\phi}\cos\theta).$$

Therefore $$T = Ma^2\dot{\phi}^2 + \tfrac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2 - 2\dot{\theta}\dot{\phi}\cos\theta). \qquad 13.8.5$$

The potential energy is

$$V = \text{constant} - mga\cos\theta. \qquad 13.8.6$$

The lagrangian equation in θ becomes

$$a(\ddot{\theta} - \ddot{\phi}\cos\theta) + g\sin\theta = 0. \qquad 13.8.7$$

The lagrangian equation in ϕ becomes

$$(2M + m)\ddot{\phi} = m(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta). \qquad 13.8.8$$

These, then, are two differential equations in the two variables. The lagrangian part of the analysis is over; we now have to see if we can do anything with these equations.

It is easy to eliminate $\ddot{\phi}$ and hence get a single differential equation in θ:

$$(2M + m\sin^2\theta)a\ddot{\theta} + ma\sin\theta\cos\theta\dot{\theta}^2 + (2M + m)g\sin\theta = 0. \qquad 13.8.9$$

If you are good at differential equations, you might be able to do something with this, and get θ as a function of the time. In the meantime, I think I can get the "first space integral" (see Chapter 6) – i.e. $\dot{\theta}$ as a function of θ. Thus, the total energy is constant:

$$Ma^2\dot{\phi}^2 + \tfrac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2 - 2\dot{\theta}\dot{\phi}\cos\theta) - mga\cos\theta = E. \qquad 13.8.10$$

Here I am measuring the potential energy from the centre of the circle. Also, if we assume that the initial condition is that at time $t = 0$ the kinetic energy was zero and $\theta = \alpha$, then $E = -mga\cos\alpha$.

Equation 13.8.8 can easily be integrated once with respect to time, since $\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta = \frac{d}{dt}(\dot{\theta}\cos\theta)$, as would have been apparent during the derivation of equation 13.8.8. With the condition that the kinetic energy was initially zero, integration of equation13.8.8 gives

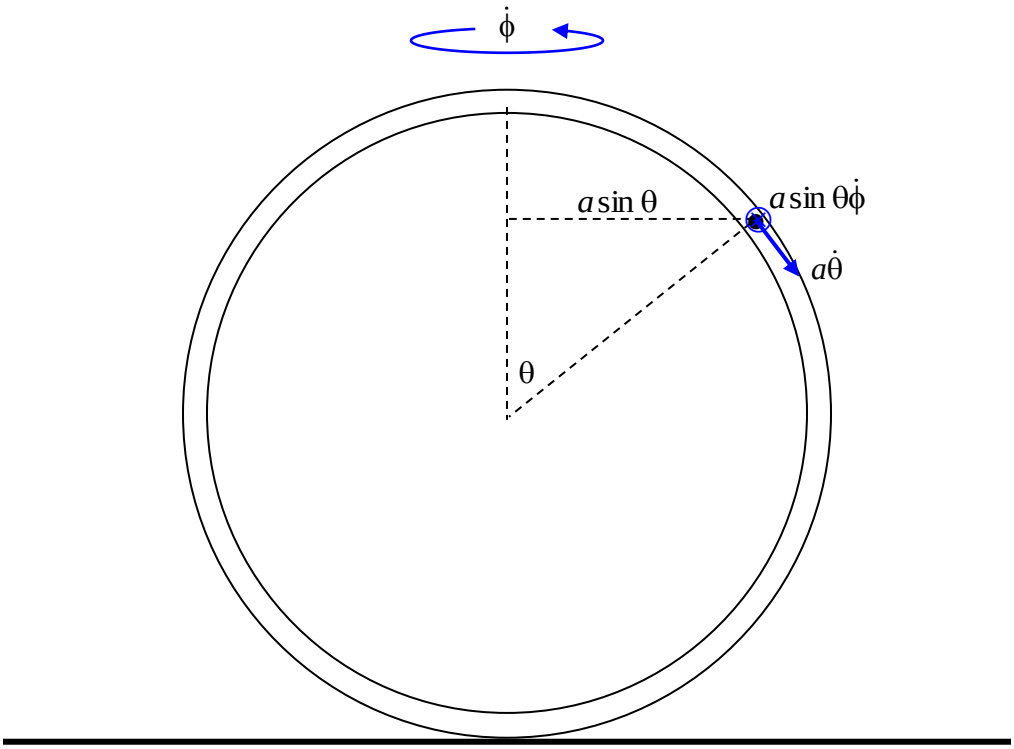
$$(2M+m)\dot{\phi} = m\dot{\theta}\cos\theta. \qquad 13.8.11$$

Now we can easily eliminate $\dot{\phi}$ between equations 13.8.10 and 11, to obtain a single equation relating $\dot{\theta}$ and θ:

$$b\dot{\theta}^2(1 + c\sin^2\theta) - d\cos\theta - 1 = 0, \qquad 13.8.12$$

where $b = \frac{Mma^2}{(2M+m)E}$, $c = \frac{m}{2M}$, $d = \frac{mga}{E} = -\sec\alpha$. 13.8.13a,b,c

iii.



As in example ii, we have a torus of radius *a* and mass *M*, and a pearl of mass *m* which can slide freely and without friction around the torus. This time, however, the torus is not rolling along the table, but is spinning about a vertical axis at an angular speed $\dot{\phi}$. The pearl has a velocity component $a\dot{\theta}$ because it is sliding around the torus, and a component $a\sin\theta\dot{\phi}$ because the torus is spinning. The resultant speed is the orthogonal

sum of these. The kinetic energy of the system is the sum of the translational kinetic energy of the pearl and the rotational kinetic energy of the torus:

$$T = \tfrac{1}{2}ma^2(\dot\theta^2 + \sin^2\theta\dot\phi^2) + \tfrac{1}{2}(\tfrac{1}{2}Ma^2)\dot\phi^2. \qquad 13.8.14$$

If we refer potential energy to the centre of the torus:

$$V = mga\cos\theta. \qquad 13.8.15$$

The lagrangian equations with respect to the two variables are

θ: $$a(\ddot\theta - \sin\theta\cos\theta\dot\phi^2) - g\sin\theta = 0. \qquad 13.8.16$$

ϕ: $$m\sin^2\theta\dot\phi + \tfrac{1}{2}M\dot\phi = \text{constant}. \qquad 13.8.17$$

The constant is equal to whatever the initial value of the left hand side was. E.g., maybe the initial values of θ and $\dot\phi$ were α and ω. This finishes the lagrangian part of the analysis. The rest is up to you. For example, it would be easy to eliminate $\dot\phi$ between these two equations to obtain a differential equation between θ and the time. If you then write $\ddot\theta$ as $\dot\theta d\dot\theta/d\theta$ in the usual way, I think it wouldn't be too difficult to obtain the first space integral and hence get $\dot\theta$ as a function of θ. I haven't tried it, but I'm sure it'll work.

iv.

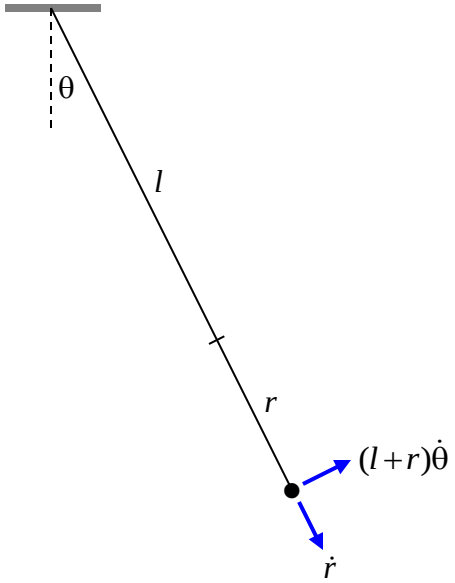


FIGURE XIII.9

Figure XIII.10 shows a pendulum. The mass at the end is *m*. It is at the end not of the usual inflexible string, but of an elastic spring obeying Hooke's law, of force constant *k*.

The spring is sufficiently stiff at right angles to its length that it remains straight during the motion, and all the motion is restricted to a plane. The unstretched natural length of the spring is *l*, and, as shown, its extension is *r*. The spring itself is "light" in the sense that it does not contribute to the kinetic or potential energies. (You can give the spring a finite mass if you want to make the problem more difficult.) The kinetic and potential energies are

$$T = \tfrac{1}{2}m\left(\dot{r}^2 + (l+r)^2\dot{\theta}^2\right) \qquad 13.8.18$$

and

$$V = \text{constant} - mg(l+r)\cos\theta + \tfrac{1}{2}kr^2. \qquad 13.8.19$$

Apply Lagrange's equation in turn to *r* and to θ and see where it leads you.

v. Another example suitable for lagrangian methods is given as problem number 11 in Appendix A of these notes.

Lagrangian methods are particularly applicable to vibrating systems, and examples of these will be discussed in chapter 17. These chapters are being written in more or less random order as the spirit moves me, rather than in logical order, so that vibrating systems appear after the unlikely sequence of relativity and hydrostatics.

13.9 *Hamilton's Variational Principle*

Hamilton's variational principle in dynamics is slightly reminiscent of the principle of virtual work in statics, discussed in section 9.4 of Chapter 9. When using the principle of virtual work in statics we imagine starting from an equilibrium position, and then increasing one of the coordinates infinitesimally. We calculate the virtual work done and set it to zero. I am slightly reminded of this when discussing Hamilton's principle in dynamics

Imagine some mechanical system – some contraption including in its construction various wheels, jointed rods, springs, elastic strings, pendulums, inclined planes, hemispherical bowls, and ladders leaning against smooth vertical walls and smooth horizontal floors. It may require *N* generalized coordinates to describe its configuration at any time. Its configuration could be described by the position of a point in *N*-dimensional space. Or perhaps it is subject to *k* holonomic constraints – in which case the point that describes its configuration in *N*-dimensional space is not free to move anywhere in that space, but is constrained to slither around on a surface of dimension $N-k$.

The system is not static, but it is evolving. It is changing from some initial state at time t_1 to some final state at time t_2. The generalized coordinates that describe it are changing

with time – and the point in N-space is slithering round on its surface of dimension $N - k$. One can imagine that at any instant of time one can calculate its kinetic energy T and its potential energy V, and hence its lagrangian $L = T - V$. You can multiply L at some moment by a small time interval δt and then add up all of these products between t_1 and t_2 to form the integral

$$\int_{t_1}^{t_2} L\,dt.$$

This quantity – of dimension ML^2T^{-1} and SI unit J s – is sometimes called the "action". There are many different ways in which we can imagine the system to evolve from its initial state to its final state – and there are many different routes that we can imagine might be taken by our point in N-space as its moves from its initial position to its final position, as long as it moves over its surface of dimension $N - k$. But, although we can *imagine* many such routes, the manner in which the system will *actually* evolve, and the route that the point will actually take is determined by Hamilton's principle; and the route, according to this principle, is such that the integral $\int_{t_1}^{t_2} L\,dt$ is a minimum, or a maximum, or an inflection point, when compared with other imaginable routes. Stated otherwise, let us suppose that we calculate $\int_{t_1}^{t_2} L\,dt$ over the actual route taken and then calculate the *variation* in $\int_{t_1}^{t_2} L\,dt$ if the system were to move over a slightly different adjacent path. Then (and here is the analogy with the principle of virtual work in a statics problem) this *variation*

$$\delta\int_{t_1}^{t_2} L\,dt$$

from what $\int_{t_1}^{t_2} L\,dt$ would have been over the actual route is zero. And this is Hamilton's variational principle.

The next questions will surely be: Can I use this principle for solving problems in mechanics? Can I prove this bald assertion?

Let me try to use the principle to solve two simple and familiar problems, and then move on to a more general problem.

The first problem will be this. Imagine that we have a particle than can move in one dimension (i.e. one coordinate – for example its height y above a table – suffices to describe its position), and that when its coordinate is y its potential energy is

$$V = mgy. \qquad 13.9.1$$

Its kinetic energy is, of course,

$$T = \tfrac{1}{2} m\dot{y}^2. \qquad 13.9.2$$

We are going to use the variational principle to find the equation of motion – i.e we are going to find an expression for its acceleration. I imagine at the moment you have no idea what its acceleration could possibly be – but don't worry, for we know that the lagrangian is

$$L = \tfrac{1}{2} m\dot{y}^2 - mgy, \qquad 13.9.3$$

and we'll make short work of it with Hamilton's variational principle and soon find the acceleration. According to this principle, y must vary with t in such a manner that

$$m\delta\int_{t_1}^{t_2}(\tfrac{1}{2}\dot{y}^2 - gy)dt = 0. \qquad 13.9.4$$

Let us vary $\dot{y}$ by $\delta\dot{y}$ and y by δy, and see how the integral varies.

The integral is then $$m\int_{t_1}^{t_2}(\dot{y}\,\delta\dot{y} - g\,\delta y)dt, \qquad 13.9.5$$

which I'll call $I_1 - I_2$.

Now $\dot{y} = \dfrac{dy}{dt}$, and if y varies by δy, the resulting variation in $\dot{y}$ will be $\delta\dot{y} = \dfrac{d}{dt}\delta y$, *or* $\delta\dot{y}\,dt = d\,\delta y$.

Therefore $$I_1 = m\int_{t_1}^{t_2}\dot{y}d\delta y. \qquad 13.9.6.$$

(If unconvinced of this, consider $\int e^t \cos t\,dt = \int e^t \dfrac{d}{dt}\sin t\,dt = \int e^t d\sin t$.)

By integration by parts: $$I_1 = [m\dot{y}\,\delta y]_{t_1}^{t_2} - m\int_{t_1}^{t_2}\delta y\,d\dot{y}. \qquad 13.9.10$$

The first term is zero because the variation is zero at the beginning and end points. In the second term, $d\dot{y} = \ddot{y}\,dt$, and therefore

$$I_1 = -m\int_{t_1}^{t_2}\ddot{y}\,\delta y\,dt. \qquad 13.9.11$$

$\therefore$ $$\delta\int_{t_1}^{t_2}L\,dt = -m\int_{t_1}^{t_2}(\ddot{y} + g)\delta y\,dt, \qquad 13.9.12$$

and, for this to be zero, we must have

$$\ddot{y} = -g\,. \qquad 13.9.13$$

This is the equation of motion that we sought. You would never have guessed this, would you?

Now let's do another one-dimensional problem. Only one coordinate, *x*, describes the particle's position, and, when its coordinate is *x* we'll suppose that its potential energy is $V = \frac{1}{2}m\omega^2 x^2$, and its kinetic energy is, of course, $T = \frac{1}{2}m\dot{x}^2$. The equation of motion, or the way in which the acceleration varies with position, must be such as to satisfy

$$\tfrac{1}{2}m\delta\int_{t_1}^{t_2}(\dot{x}^2 - \omega^2 x^2)dt = 0. \qquad 13.9.14$$

If we vary $\dot{x}$ by $\delta\dot{x}$ and x by δx, the variation in the integral will be

$$m\int_{t_1}^{t_2}(\dot{x}\,\delta\dot{x} - \omega^2 x\delta x)dt = I_1 - I_2, \text{ say.} \qquad 13.9.15$$

By precisely the same argument as before, the first integral is found to be $-m\int_{t_1}^{t_2}\ddot{x}\,\delta x\,dt$.

Therefore $$\delta\int_{t_1}^{t_2}L\,dt = -m\int_{t_1}^{t_2}\ddot{x}\,\delta x\,dt - m\omega^2\int_{t_1}^{t_2}x\,\delta x\,dt, \qquad 13.9.16$$

and, for this to be zero, we must have

$$\ddot{x} = -\omega^2 x\,. \qquad 13.9.17$$

These two examples must have given the impression that we are doing something very difficult in order to derive something that is immediately obvious – but the examples were just intended to show the direction of a more general argument we are about to make.

This time, we'll consider a very general system, in which we write the lagrangian as a function of the (several) generalized coordinates and their time rates of change - i.e. $L = L(q_i, \dot{q}_i)$ - without specifying any particular form of the function – and we'll carry out the same sort of argument to derive a very general equation of motion.

We have $$\delta\int_{t_1}^{t_2}L\,dt = \int_{t_1}^{t_2}\delta L\,dt = \int_{t_1}^{t_2}\sum_i\left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i\right)dt = 0. \qquad 13.9.18$$

As before, $\delta\dot{q}_i = \dfrac{d}{dt}\delta q_i$, so that

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i\,dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i\,dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i}d\delta q_i = \left[\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right]_{t_i}^{t_2} - \int_{t_1}^{t_2}\delta q_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}dt. \quad 13.9.19$$

$$\therefore \qquad \delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\sum_i\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\,dt = 0. \quad 13.9.20$$

Thus we arrive at the general equation of motion

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0. \quad 13.9.21$$

Thus we have derived Lagrange's equation of motion from Hamilton's variational principle, and this is indeed the way it is often derived. However, in this chapter, I derived Lagrange's equation quite independently, and hence I would regard this derivation not so much as a proof of Lagrange's equation but as a vindication of the correctness of Hamilton's variational principle.