CHAPTER 14
GENERAL PERTURBATION THEORY

14.1 *Introduction*

A particle in orbit around a point mass – or a spherically symmetric mass distribution – is moving in a gravitational potential of the form $-GM/r$. In this potential it moves in a keplerian ellipse (or hyperbola if its kinetic energy is large enough) that can be described by the six orbital elements *a*, *e*, *i*, Ω, ω, *T*, or any equivalent set of six parameters.

If the potential is a little different from $-GM/r$, say $-(GM/r \ + \ R)$, the orbit will be *perturbed*, and *R* is described as a *perturbation*. As a result it will no longer move in a perfect keplerian ellipse. Perturbations may be *periodic* or *secular*. For example, the elements such as *a*, *e* or *i* may vary in a periodic fashion, while there may be *secular* changes (i.e. changes that are not periodic but constantly increase or decrease in the same direction) in elements such as Ω and ω. (That is, the line of nodes and the line of apsides may monotonically precess; they may *advance* or *regress*.)

In some situations it may be possible to express the perturbation in terms of a simple algebraic formula. An example would be a particle in orbit around a slightly oblate planet, where it is possible to express the potential algebraically. The aim of this chapter will be to try to find general expressions for the rates of change of the orbital elements in terms of the perturbing function, and we shall use the orbit around an oblate planet as an example.

In other situations it is not easily possible to express the perturbation in terms of a simple algebraic function. For example, a planet in orbit around the Sun is subject not only to the gravitational field of the Sun, but to the perturbations caused by all the other planets in the solar system. These special perturbations have to be treated numerically, and the techniques for doing so will be described in chapter 15.

14.2 *Contact Transformations and General Perturbation Theory*

(Before reading this section, it may be well to re-read section 10.11 of Chapter 10.)

Suppose that we have a simple problem in which we know the hamiltonian H_0 and that the Hamilton-Jacobi equation has been solved:

$$H_0\left(q_1\ ,\frac{\partial S}{\partial q_1}\ ,\ t\right)\ +\ \frac{\partial S}{\partial t}\ =\ 0. \qquad 14.2.1$$

Now suppose we have a similar problem, but that the hamiltonian, instead of being just H_0 is $H = H_0 - R$, and $K = H + \frac{\partial S}{\partial t}\cdot$

Let us make a contact transformation from (p_i, q_i) to (P_i, Q_i), where $\dot{Q}_i = \frac{\partial K}{\partial P_i}$ and $\dot{P}_i = -\frac{\partial K}{\partial Q_i}\cdot$ In the orbital context, following Section 10.11, we identify Q_i with α_i and P_i with $-\beta_i$, which are functions (given in Section 10.11) of the orbital elements and which can serve in place of the orbital elements. The parameters are constants with respect to the unperturbed problem, but are *variables* with respect to the perturbing function. They are given, as functions of time, by the solution of Hamilton's equations of motion, which retain their form under a contact transformation.

$$\dot{\alpha}_i = \frac{\partial R}{\partial \beta_i} \quad \text{and} \quad \dot{\beta}_i = -\frac{\partial R}{\partial \alpha_i}. \qquad 14.2.2\text{a,b}$$

Perturbation theory will show, then, how the α_i and β_i will vary with a given perturbation. The conventional elements a, e, i, Ω, ω, T are functions of α_i, β_i, and our aim is to find how the conventional elements vary with time under the perturbation R.

We can do that as follows. Let A_i be an orbital element, given by

$$A_i = A_i(\alpha_i, \beta_i). \qquad 14.2.3$$

Then
$$\dot{A}_i = \sum_j \frac{\partial A_i}{\partial \alpha_j}\dot{\alpha}_j + \sum_j \frac{\partial A_i}{\partial \beta_j}\dot{\beta}_j . \qquad 14.2.4$$

By equations 14.2.2a,b, this becomes

$$\dot{A}_i = \sum_j \frac{\partial A_i}{\partial \alpha_j}\frac{\partial R}{\partial \beta_j} - \sum_j \frac{\partial A_i}{\partial \beta_j}\frac{\partial R}{\partial \alpha_j} . \qquad 14.2.5$$

But
$$\frac{\partial R}{\partial \alpha_j} = \sum_k \frac{\partial R}{\partial A_k}\frac{\partial A_k}{\partial \alpha_j} \quad \text{and} \quad \frac{\partial R}{\partial \beta_j} = \sum_k \frac{\partial R}{\partial A_k}\frac{\partial A_k}{\partial \beta_j}. \qquad 14.2.6\text{a,b}$$

$\therefore$
$$\dot{A}_i = \sum_j \sum_k \frac{\partial R}{\partial A_k}\left(\frac{\partial A_i}{\partial \alpha_j}\frac{\partial A_k}{\partial \beta_j} - \frac{\partial A_i}{\partial \beta_j}\frac{\partial A_k}{\partial \alpha_j}\right) \qquad 14.2.6$$

That is $$\dot{A}_i = \sum_k \frac{\partial R}{\partial A_k} \sum_j \left(\frac{\partial A_i}{\partial \alpha_j}\frac{\partial A_k}{\partial \beta_j} - \frac{\partial A_i}{\partial \beta_j}\frac{\partial A_k}{\partial \alpha_j} \right). \qquad 14.2.7$$

This can be written, in shorthand:

$$\dot{A}_i = \sum_k \frac{\partial R}{\partial A_k} \{A_i\, ,\, A_k\}_{\alpha_j,\beta_j}\,. \qquad 14.2.8$$

Here the symbol $\{A_i\, ,\, A_k\}_{\alpha_j,\beta_j}$ is called the *Poisson bracket* of A_i, A_k with respect to α_j, β_j. (In the language of the typographer, the symbols (), [] and {} are, respectively, parentheses, brackets and braces; you may refer to Poisson braces if you wish, but the usual term, in spite of the symbols, is Poisson bracket.)

Note the property $\{A_i\, ,\, A_k\}_{\alpha_j,\beta_j} = -\{A_k\, ,\, A_i\}_{\alpha_j,\beta_j}$.

14.3 *The Poisson Brackets for the Orbital Elements*

A worked example is in order. From equations 14.2.7 and 14.2.8, we see that the Poisson brackets are defined by

$$\{A_i\, ,\, A_k\}_{\alpha_j,\beta_j} = \sum_j \left(\frac{\partial A_i}{\partial \alpha_j}\frac{\partial A_k}{\partial \beta_j} - \frac{\partial A_i}{\partial \beta_j}\frac{\partial A_k}{\partial \alpha_j} \right). \qquad 14.3.1$$

The A_i are the orbital elements.

For our example, we shall calculate $\{\Omega\, ,\, i\}$ and we write out the sum in full:

$$\{\Omega\, ,\, i\} = \sum_j \left(\frac{\partial \Omega}{\partial \alpha_j}\frac{\partial i}{\partial \beta_j} - \frac{\partial \Omega}{\partial \beta_j}\frac{\partial i}{\partial \alpha_j} \right)$$

$$= \frac{\partial \Omega}{\partial \alpha_1}\frac{\partial i}{\partial \beta_1} + \frac{\partial \Omega}{\partial \alpha_2}\frac{\partial i}{\partial \beta_2} + \frac{\partial \Omega}{\partial \alpha_3}\frac{\partial i}{\partial \beta_3} - \frac{\partial \Omega}{\partial \beta_1}\frac{\partial i}{\partial \alpha_1} - \frac{\partial \Omega}{\partial \beta_2}\frac{\partial i}{\partial \alpha_2} - \frac{\partial \Omega}{\partial \beta_3}\frac{\partial i}{\partial \alpha_3}\,. \qquad 14.3.2$$

Refer now to equations 10.11.27 and 29, and we find

$$\{\Omega\, ,\, i\} = 0 + 0 + 0 - 0 + \frac{1}{\alpha_3\sqrt{1-\alpha_2^2/\alpha_3^2}} - 0\,. \qquad 14.3.3$$

Finally, referring to equations 10.11.20 and 21, we obtain

$$\{\Omega\ ,\ i\}\ =\ \frac{1}{\sqrt{GMm^2a(1-e^2)}\,.\sin i}\ . \qquad 14.3.4$$

Proceeding in a similar manner for the others, we obtain

$$\{a\ ,T\}\ =\ -\frac{2a^2}{GMm}\ , \qquad 14.3.5$$

$$\{e\ ,T\}\ =\ -\frac{a(1-e^2)}{GMme}\ , \qquad 14.3.6$$

$$\{e,\omega\}\ \ =\ -\frac{\sqrt{1-e^2}}{em\sqrt{GMa}} \qquad 14.3.7$$

$$\{i\ ,\ \omega\}\ =\ \frac{1}{\sqrt{GMm^2a(1-e^2)}\,.\tan i}\ . \qquad 14.3.8$$

In addition, we have, of course,

$$\{i\ ,\ \Omega\}\ =\ -\{\Omega\ ,\ i\}\ ,\ \{T\ ,\ a\}\ =\ -\{a\ ,\ T\}\ ,\ \{T\ ,\ e\}\ =\ -\{e\ ,\ T\},$$
$$\{e,\,\omega\}=-\{\omega,e\}\ \text{ and }\ \{\omega\ ,\ i\}\ =\ -\{i\ ,\ \omega\}.$$

All other pairs are zero.

14.4 *Lagrange's Planetary Equations*

We now go to equation 14.2.8 to obtain *Lagrange's Planetary Equations*, which will enable us to calculate the rates of change of the orbital elements if we know the form of the perturbing function:

$$\dot{a}\ =\ -\frac{2a^2}{GMm}\ \frac{\partial R}{\partial T}\ , \qquad 14.4.1$$

$$\dot{e}\ =\ -\frac{a(1-e^2)}{GMme}\ \frac{\partial R}{\partial T}\ -\ \frac{1}{me}\sqrt{\frac{1-e^2}{GMa}}\ \frac{\partial R}{\partial \omega}\ , \qquad 14.4.2$$

$$\dot{i}\ =\ -\frac{1}{\sqrt{GMm^2a(1-e^2)}\sin i}\ \frac{\partial R}{\partial \Omega}\ +\frac{1}{\sqrt{GMm^2a(1-e^2)}\tan i}\ \frac{\partial R}{\partial \omega}\ , \qquad 14.4.3$$

$$\dot{\omega} = \frac{1}{me}\sqrt{\frac{1-e^2}{GMa}}\;\frac{\partial R}{\partial e} - \frac{1}{\sqrt{GMm^2a(1-e^2)}\tan i}\;\frac{\partial R}{\partial i}\,, \qquad 14.4.4$$

$$\dot{\Omega} = \frac{1}{\sqrt{GMm^2(1-e^2)}\sin i}\;\frac{\partial R}{\partial i}\,, \qquad 14.4.5$$

$$\dot{T} = \frac{2a^2}{GMm}\;\frac{\partial R}{\partial a} + \frac{a(1-e^2)}{GMme}\;\frac{\partial R}{\partial e}\,. \qquad 14.4.6$$

14.5 *Motion Around an Oblate Symmetric Top*

In Section 5.12 we developed an expression (equation 5.12.6) for the gravitational potential of an oblate symmetric top (e.g. an oblate spheroid). With a slight change of notation to conform to the present context, we obtain for the perturbing function

$$R = \frac{Gm(C-A)}{2r^3}\left(1 - \frac{3z^2}{r^2}\right). \qquad 14.5.1$$

This is the negative of the additional potential *energy* of a mass m at a point whose cylindrical coordinates are $(r\,,\,z)$ in the vicinity of a symmetric top (which I'll henceforth call an oblate spheroid) whose principal second moments of inertia are C (polar) and A (equatorial). This is correct to order r/a, where a is the equatorial radius of the spheroid.

Let us imagine a particle of mass m in orbit around an oblate spheroid – e.g. an artificial satellite in orbit around Earth. Suppose the orbit is inclined at an angle i to the equator, and the argument of perigee is ω. At some instant, when the cylindrical coordinates of the satellite are $(r\,,\,z)$, its true anomaly is υ.

Exercise (in geometry): Show that $z/r = \sin i \sin(\omega + \upsilon)$.

Having done that, we see that the perturbing function can be written

$$R = \frac{Gm(C-A)}{2r^3}\left(1 - 3\sin^2 i \sin^2(\omega + \upsilon)\right). \qquad 14.5.2$$

Here, r and υ vary with time, or what amounts to the same thing, with the mean anomaly $\boldsymbol{M}$. With a (nontrivial) effort, this can be expanded as a series, including a constant (time independent) term plus periodic terms of the form $\cos \boldsymbol{M}$, $\cos 2\boldsymbol{M}$, $\cos 3\boldsymbol{M}$, etc. If the spirit moves me, I may post the details at a later date, but for the present I give the result that, if the expansion is taken as far as e^2 (i.e. we are assuming that the orbit of the satellite is not strongly eccentric), the constant (time-independent) part of the perturbing function is

$$R = \frac{Gm(C - A)}{2a^3}(1 + \tfrac{3}{2}e^2)(1 - \tfrac{3}{2}\sin^2 i). \qquad 14.5.3$$

Now look at Lagrange's equations, and you see that the secular parts of $\dot{a}$, $\dot{e}$ and $\dot{i}$ are all zero. That is, although there may be periodic variations (which we have not examined) in these elements, to this order of approximation (e^2) there is no secular change in these elements.

On the other hand, application of equation 14.4.5 gives for the secular rate of change of the longitude of the nodes

$$\dot{\Omega} = -\frac{3\sqrt{GM}}{2}\frac{(C - A)}{M}\frac{1}{a^{7/2}}(1 + 2e^2)\cos i. \qquad 14.5.4$$

The reader will no doubt be relieved to note that this expression does not contain m, the mass of the orbiting satellite; M is the mass of the Earth. The reader may also note the minus sign, indicating that the nodes *regress*. To obtain the factor $(1 + 2e^2)$, readers will have to do a little bit of work, and to expand, by the binomial theorem, whatever expression in e they get, as far as e^2.

Let a be the equatorial radius of Earth. Multiply top and bottom of equation 14.5.4 by $\mathsf{a}^{7/2}$, and the equation becomes

$$\dot{\Omega} = -\frac{3}{2}\sqrt{\frac{GM}{\mathsf{a}^3}}\frac{(C - A)}{M\mathsf{a}^2}\left(\frac{\mathsf{a}}{a}\right)^{7/2}(1 + 2e^2)\cos i. \qquad 14.5.5$$

Here M is the mass of Earth (not of the orbiting satellite), a is the semi major axis of the satellite's orbit, and a is the equatorial radius of Earth.

[If we assume Earth is an oblate spheroid of uniform density, then, according to example 1.iii of Section 2.20 of Chapter 2 of our notes on Classical Mechanics, $C = \tfrac{2}{5}M\mathsf{a}^2$. In that case, equation 14.5.5 becomes $\dot{\Omega} = -\frac{3}{5}\sqrt{\frac{GM}{\mathsf{a}^3}}\frac{(C - A)}{C}\left(\frac{\mathsf{a}}{a}\right)^{7/2}(1 + 2e^2)\cos i.$ But the density of Earth is not uniform, so we'll leave equation 14.5.5 as it is.]

For a nearly circular orbit, equation 14.5.5 becomes just

$$\dot{\Omega} = -\frac{3}{2}\sqrt{\frac{GM}{\mathsf{a}^3}}\frac{(C - A)}{M\mathsf{a}^2}\left(\frac{\mathsf{a}}{a}\right)^{7/2}\cos i. \qquad 14.5.6$$

This tells us that the line of nodes of a satellite in orbit around an oblate planet (i.e $C > A$) *regresses*. From the rate of regression of the line of nodes, we can deduce the difference, $C - A$ between the principal moments of inertia, though we cannot deduce either moment separately. (If we *could* determine the moment of inertia from the rate of regression of the nodes – which we cannot – how well can we determine the density distribution inside Earth? See Problem 14 in Chapter A of our Classical Mechanics notes to determine the answer to this. It will be found that knowledge of the moment of inertia places only weak constraints on the core size and density.)

Numerically it is known for Earth that the quantity $\frac{3}{2}\sqrt{\frac{GM}{\mathrm{a}^3}}\frac{(C-A)}{M\mathrm{a}^2}$ is about 2.04 rad s^{-1}, or about 10.1 degrees per day. Thus the rate of regression of the nodes of a satellite in orbit around Earth in a near-circular orbit is about

$$\dot{\Omega} = -10.1\left(\frac{\mathrm{a}}{a}\right)^{7/2}\cos i \qquad \text{degrees per day.}$$

We can refer to equations 14.4.4 and 14.5.3 to determine the rate of motion of the line of apsides, $\dot{\omega}$. After some algebra, and neglect of terms of order e^2 and higher, we find

$$\dot{\omega} = \frac{3(C-A)}{4a^{7/2}}\sqrt{\frac{G}{M}}(5\cos^2 i - 1), \qquad 14.5.7$$

or, if we multiply top and bottom by $\mathrm{a}^{7/2}$,

$$\dot{\omega} = \frac{3(C-A)}{4M\mathrm{a}^2}\sqrt{\frac{GM}{\mathrm{a}^3}}\left(\frac{\mathrm{a}}{a}\right)^{7/2}(5\cos^2 i - 1). \qquad 14.5.8$$

Thus we find that the line of apsides advances if the inclination of the orbit to the equator is less than 63° and it regresses if the inclination is greater than this.

In this section, I have demanded a fair amount of work from the reader – in particular for the expansion of equation 14.5.2. While the work requires some patience and persistence, it is straightforward, and the resolute reader will be able to work out the expansion in terms of the mean anomaly and the time, and hence, by making use of Lagrange's planetary equations, will be able to predict the periodic variations in a, e and i. For the time being, I am not going to do this, since no new principles are involved, the aim of the chapter being to give the reader a start on how to start to calculate the changes in the orbital elements if one can express the perturbing function analytically.

For the effect of the perturbation of a planetary orbit by the presence of other planets, we have to solve the problem numerically by the techniques of *special perturbations*, which, I hope, some time in the future, may be the subject of an additional chapter.